

AN ANALYSIS OF A COMPOUND  
PENDULUM ROCKET SUSPENSION

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JAMES CASPAR NORRIS, JR.

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AN ANALYSIS OF A COMPOUND  
PENDULUM ROCKET SUSPENSION

Thesis by

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## SUMMARY

This is an investigation of the equations of motion and physical parameters involved in stabilizing the initial flight of a vertically launched rocket by means of a booster rocket pin-connected below the main rocket. The system is designed to stabilize the flight in its early stage before the aerodynamic control surfaces become effective. Stability of the system is dependent on the pendulum action of the booster rocket.

The equations of motion were derived from Lagrange's generalized momentum equation. The differential equations thus obtained were not solved but were tested for stability by means of Routh's stability criteria. The ratio of the mass of the main rocket,  $M_1$ , to the mass of the booster rocket,  $M_2$ , was investigated for the two values  $\frac{M_1}{M_2} = 1.5$  and  $\frac{M_1}{M_2} = 7.75$ .

The system involving a mass ratio  $\frac{M_1}{M_2} = 7.75$  was found to be unstable under all conditions. However, the system involving a mass ratio  $\frac{M_1}{M_2} = 1.5$  was determined to be stable in the range  $1.62 < \beta < 4.54 \times 10^{10}$ , where  $\beta$  is defined as the ratio of the distance  $l_2$  from the center of gravity of the booster  $M_2$  to the pin connecting the strut to the main rocket  $M_1$ , divided by the radius of gyration,  $k_2$ , of the booster  $M_2$ . In this range, for any given value of  $\beta$ , stability was uniquely determined by



one value of the ratio  $\alpha = \frac{l_1}{l_2}$ , where  $l_1$  is the length of the strut from the main rocket  $M_1$  to the booster rocket  $M_2$ . Thus, for a given booster, stability is primarily a function of the ratio  $\frac{l_1}{l_2}$ , and for any given  $l_2$ ,  $l_1$  is uniquely determined.

Although the system was found to be theoretically stable for the mass ratio  $\frac{M_1}{M_2} = 1.5$ , the ratio  $\frac{l_1}{l_2}$  turned out to be of such great magnitude as to make the system entirely impractical for this particular mass ratio.



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## I. INTRODUCTION

When a rocket or guided missile is initially launched there is a short period when its flight is unstable due to the fact that at low velocities the aerodynamic control surfaces are ineffective.

Various procedures have been adopted to stabilize this initial part of the flight. The most obvious and most successful of these has been the use of guide rails, or launching towers which hold the missile on its course until it has attained a velocity at which the control surfaces become effective. But this method requires a bulky launcher, and special care is necessary to maintain the rails in proper alignment.

A modification of the guide rail system is the "short-length" launcher which is really a guide rail whose length has been reduced to a minimum by greatly increasing the initial acceleration and thus the velocity of the missile. This increased acceleration presents a great many problems in component design to resist the tremendous accelerative forces.

During World War II the Germans were fairly successful with a unique approach to the problem in their well-known "V-2" rocket. (Reference 1). The "V-2" was launched from a near-vertical position with no external restrictions. Instead it employed four carbon vanes mounted in the jet stream and activated by the auto-pilot to maintain the missile on its proper flight path.



The great difficulty here was that the vanes steadily burned out with a consequent diminishing of control.

The subject of this thesis is a third means of initial flight stabilization based on the concept of the action of a pendulum and independent of both external guides and internal control mechanisms.

If the missile booster is constructed as a separate unit and attached to the main rocket by a pin connection a certain distance below its own center of gravity it can be seen from Figure 1 that any motion of the main rocket will impart a proportionate motion to the booster. Specifically, if the main rocket rotates about its center of gravity in a clockwise direction, the connecting strut will cause the booster to rotate also about its own center of gravity in a clockwise direction. Thus, if the main rocket turns off course in a clockwise direction the booster immediately turns in a clockwise direction also, that is, the booster tends to re-align its thrust with the axis of the main rocket. If, by a proper choice of moment arms, the booster can be made to overshoot this position of thrust alignment slightly the thrust will have a horizontal component which acts on the strut to the main rocket and causes the main rocket to turn about its center of gravity in a counter-clockwise direction. Thus, the booster responds to an error and its response tends to counteract the error. Such a sys-



tem should produce a sinusoidal flight path if the proper moment arms and damping forces were applied.

In this analysis frictional and aerodynamic forces were neglected to simplify the equations. The only damping force considered (other than the inertia forces of the system) was the jet damping force or the resistance of a jet stream to rotation. The other external forces considered were the weights of the main rocket and the booster, and the thrust of the booster. (The motor of the main rocket is considered not to be operating in this analysis). Taking into account only these forces the writer derived the equations of motion of the system in two dimensions by means of Lagrange's generalized momentum equation. Because of the complicated nature of these equations no solution of them was attempted. Instead the conditions for stability were investigated by means of Routh's stability criteria.

A hypothetical system consisting of a five-second booster pin-connected below a "V-2" rocket was first investigated. For this system the ratio of the mass of the main rocket  $M_1$ , to the mass of the booster  $M_2$ , is  $\frac{M_1}{M_2} = 7.75$ . To reduce the work of computation the lateral motion of the center of gravity of the system was set equal to zero and only the vertical motion of the center of gravity and the rotation of the main rocket and the booster were considered.



Since no stable solution was found, a system with mass ratio  $\frac{M_1}{M_2} = 1.5$  was next investigated, again neglecting the lateral motion of the center of gravity of the system. In this case a range of stability was determined but it was of such a sensitive nature as to make impractical further investigation for this mass ratio in the general case with lateral motion taken into account.





## II. DERIVATION OF THE EQUATIONS OF MOTION

In deriving the equations of motion, only four external forces are considered, namely, the weight,  $M_1g$ , of the main rocket, the weight,  $M_2g$ , of the booster, the thrust,  $F$ , of the booster, and the jet damping force,  $D$ , of the booster. These forces are indicated in Figure 1.



Figure 1. External Forces on Main Rocket and Booster.



The jet damping force,  $D$ , is a force which resists the rotation of the booster about its own center of gravity. If the booster is of mass  $M_2$ , then the rate

of flow through the nozzle may be written as  $\frac{dM_2}{dt}$ , or  $\dot{M}_2$ . As the booster rotates about its center of gravity the tangential velocity at the nozzle exit will be

$l_s \frac{dq_y}{dt}$  or  $l_s \dot{q}_y$ . Therefore

the damping force,  $D$ , will be

equal to the rate of change of momentum, or  $\dot{M}_2 \cdot (l_s \dot{q}_y)$ . This expression has the dimensions  $\frac{M}{T} \cdot L \cdot \frac{1}{T}$ , or  $\frac{ML}{T^2}$ .

Frictional forces and aerodynamic forces are neglected in this analysis.

Both the main rocket,  $M_1$ , and the booster,  $M_2$ , are assigned three degrees of freedom, namely, lateral, vertical, and rotational motion. Note, also, that only the two dimensional case is being treated. The coordinates of masses  $M_1$  and  $M_2$  are indicated in Figure 2. The connecting strut between  $M_1$  and  $M_2$  is assumed to be of zero mass and of infinite stiffness for purposes of this analysis, and consequently the coordinates of the center of gravity of the system are indicated on the straight line joining  $M_1$  and  $M_2$  in Figure 2.





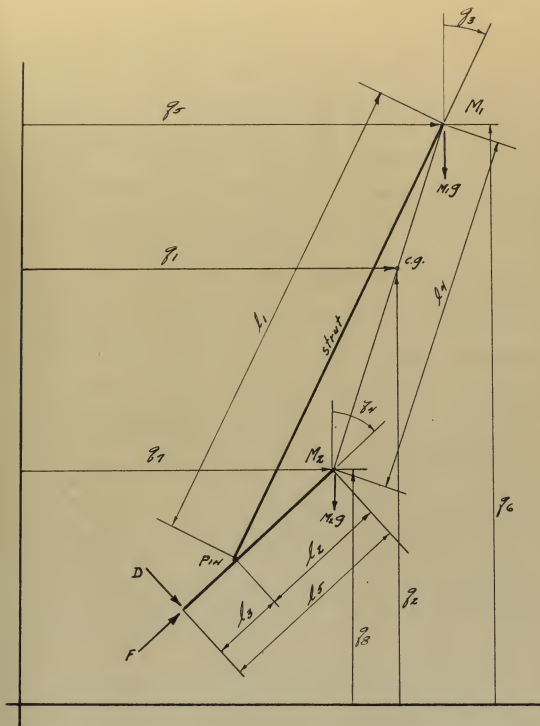


Figure 2. Schematic Diagram of Coordinate System and Dimensions.



It is assumed that the propulsion system of the main rocket,  $M_1$ , will not be operating during the initial stages of launching and consequently  $M_1$  is a constant for this analysis. The booster,  $M_2$ , will of course be burning with a consequent change of mass during launching, and the rate of change of mass  $M_2$ , say  $\dot{M}_2$ , will be considered, as will be the change of position relative to  $M_1$  and  $M_2$  of the center of gravity of the system.

The first step of the analysis is to eliminate the extraneous coordinates  $q_5$ ,  $q_6$ ,  $q_7$ , and  $q_8$  in order to write the equations of motion in terms of the rectilinear coordinates of the center of gravity of the system,  $q_1$ , and  $q_2$ , and the angular coordinates  $q_3$  and  $q_4$  of  $M_1$  and  $M_2$  about vertical axes through their respective centers of gravity. (Since only the onset of instability is of interest, the analysis will be restricted to consideration of small variations of the angles  $q_3$  and  $q_4$ .) Once this step has been completed, Lagrange's generalized momentum equation will be used to determine the equations of motion.

#### ELIMINATION OF EXTRANEIOUS COORDINATES

The coordinates  $q_5$ ,  $q_6$ ,  $q_7$ , and  $q_8$  are always redundant because of the rigid links in the system and the known position of the center of gravity of the two masses. Therefore these extraneous coordinates may be eliminated by considering the constraints





elaborated below.

The first relation between the coordinates of the individual centers of gravity and the center of gravity of the system may be found by taking moments about the center of gravity of the system indicated in Figure 3.

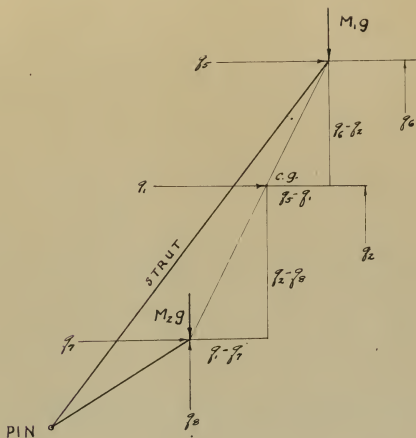


Figure 3. Relationship of  $M_1$  and  $M_2$  to the center of gravity of the system.

Thus  $M_1 g (q_5 - q_1) - M_2 g (q_1 - q_2) = 0$

or  $M_1 (q_5 - q_1) = M_2 (q_1 - q_2) \quad (1)$







A fourth equation follows from a consideration of the similar triangles indicated in Figure 3.

$$\frac{q_6 - q_2}{q_5 - q_1} = \frac{q_2 - q_8}{q_1 - q_7}$$

$$\frac{M_1(q_6 - q_2)}{M_1(q_5 - q_1)} = \frac{M_2(q_2 - q_8)}{M_2(q_1 - q_7)}$$

and, since the denominators of both sides of this equation are equal from Equation (1), it may be written

$$M_1(q_6 - q_2) = M_2(q_2 - q_8) \quad (4)$$

Equations (1), (2), (3), and (4) provide four linear relations between the coordinates  $q_1$ ,  $q_2$ ,  $q_7$ , and  $q_8$ , and  $q_1$ ,  $q_2$ ,  $q_7$ , and  $q_8$ . Consequently the former may be expressed in terms of the latter and thereby eliminated from subsequent equations. Solving these equations the following expressions are obtained:

$$\text{From Equation (2)} \quad q_6 = l_4 \cos(q_3 - \delta) + q_8 \quad (5)$$

$$\text{From Equation (3)} \quad q_5 = l_4 \sin(q_3 - \delta) + q_7$$

$$\text{or} \quad q_7 = q_5 - l_4 \sin(q_3 - \delta) \quad (6)$$

Substituting Equation (6) into Equation (1)



$$M_1(q_3 - q_1) = M_2 [q_1 - q_3 + l_4 \sin(q_3 - \delta)]$$

$$q_3 (M_1 + M_2) = M_1 q_1 + M_2 q_1 + M_2 l_4 \sin(q_3 - \delta)$$

$$q_3 = q_1 + \frac{M_2}{M_1 + M_2} l_4 \sin(q_3 - \delta) \quad (7)$$

Substituting Equation (5) into Equation (4)

$$M_1 [l_4 \cos(q_3 - \delta) + q_3 - q_2] = M_2 (q_2 - q_3)$$

$$q_3 (M_1 + M_2) = M_1 q_2 + M_2 q_2 - M_1 l_4 \cos(q_3 - \delta)$$

$$q_3 = q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos(q_3 - \delta) \quad (8)$$

Substituting Equation (7) into Equation (6)

$$q_1 = q_1 + \frac{M_2}{M_1 + M_2} l_4 \sin(q_3 - \delta) - l_4 \sin(q_3 - \delta)$$

$$q_1 = q_1 + l_4 [\sin(q_3 - \delta)] \left( \frac{M_2}{M_1 + M_2} - 1 \right)$$

$$q_1 = q_1 - \frac{M_1}{M_1 + M_2} l_4 \sin(q_3 - \delta) \quad (9)$$

From Equation (2)

$$q_2 = l_4 \cos(q_3 - \delta) + q_3$$

Substituting Equation (8) into this equation





$$g_1 = l_4 \cos(g_3 - \delta) + g_2 - \frac{M_1}{M_1 + M_2} l_4 \cos(g_3 - \delta)$$

$$g_1 = g_2 + l_4 [\cos(g_3 - \delta)] \left(1 - \frac{M_1}{M_1 + M_2}\right)$$

$$g_1 = g_2 + \frac{M_2}{M_1 + M_2} l_4 \cos(g_3 - \delta) \quad (10)$$

These expressions for the extraneous coordinates in Equations (7), (8), (9), and (10) involve, in addition to the principal coordinates, the variables  $l_4$ , the distance between the centers of gravity of the two components, and  $\delta$ , the angle between the strut  $l_1$ , and the line  $l_4$ , joining the centers of gravity of the two components. These two variables will now also be expressed in terms of the principal coordinates.

In Figure 4 by the law of cosines

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos(g_4 - g_3)} \quad (11)$$

and by the law of sines

$$\frac{l_2}{\sin \delta} = \frac{l_4}{\sin(g_4 - g_3)}$$

$$\sin \delta = \frac{l_2}{l_4} \sin(g_4 - g_3)$$

$$\delta = \sin^{-1} \left[ \frac{l_2}{l_4} \sin(g_4 - g_3) \right]$$

$$\text{whence } (g_3 - \delta) = g_3 - \sin^{-1} \left[ \frac{l_2}{l_4} \sin(g_4 - g_3) \right] \quad (12)$$



These relations are unfortunately complicated. Therefore, to simplify them for later use, the fact that the variations in  $q_1$  and  $q_2$  are small will be taken into account. Assuming these angles sufficiently small such that terms of order  $q^2$  may be neglected, the following approximations can be made

$$\sin q_i \approx q_i \quad \text{and} \quad \cos q_i \approx 1$$

From Equation (11)

$$l_4 \approx \sqrt{l_1^2 + l_2^2 - 2l_1l_2} = l_1 - l_2$$

From Equation (12)

$$(q_3 - \delta) \approx q_3 - \frac{l_2}{l_4} (q_4 - q_3)$$

$$(q_3 - \delta) \approx q_3 \left(1 + \frac{l_2}{l_4}\right) - \frac{l_2}{l_4} q_4$$

$$(q_3 - \delta) \approx q_3 \frac{l_4 + l_2}{l_4} - \frac{l_2}{l_4} q_4$$

and since  $l_4 \approx l_1 - l_2$

$$(q_3 - \delta) = \frac{l_1}{l_1 - l_2} q_3 - \frac{l_2}{l_1 - l_2} q_4 \approx \sin(q_3 - \delta)$$

$$\cos(q_3 - \delta) \approx 1$$

Thus the appropriate approximations are

$$\cos(q_3 - \delta) \approx 1$$

$$\sin(q_3 - \delta) \approx \frac{l_1}{l_1 - l_2} q_3 - \frac{l_2}{l_1 - l_2} q_4$$

$$l_4 \approx l_1 - l_2$$

} (13)



Substitution of the results of Equation (13) into Equations (7), (8), (9), and (10) yields the following approximate relations between the extraneous and the principal coordinates

$$g_3 \approx g_1 + \frac{M_2}{M_1 + M_2} (l_1 g_3 - l_2 g_4) \quad (14)$$

$$g_4 \approx g_2 + \frac{M_2}{M_1 + M_2} (l_1 - l_2) \quad (15)$$

$$g_3 \approx g_1 - \frac{M_1}{M_1 + M_2} (l_1 g_3 - l_2 g_4) \quad (16)$$

$$g_4 \approx g_2 - \frac{M_1}{M_1 + M_2} (l_1 - l_2) \quad (17)$$

To derive the equations of motion by the method of Lagrange the expression for the kinetic energy,  $T$ , must first be found for use in the Lagrangian equation

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i \quad (18)$$

where  $Q_i$  is the "generalized force" and  $\dot{q}_i$  is  $\frac{dq_i}{dt}$ . (Reference 2). The generalized force  $Q_i$  is employed rather than the potential energy because this is a non-conservative system.

If  $I_c$  denotes the moment of inertia of the mass  $M_2$  about its own center of gravity, the kinetic energy of translation and rotation for the system may be written



$$T = \frac{1}{2} I_1 \dot{\theta}_s^2 + \frac{1}{2} I_2 \dot{\theta}_r^2 + \frac{1}{2} M (\dot{\theta}_s^2 + \dot{\theta}_r^2) + \frac{1}{2} M_2 (\dot{\theta}_r^2 + \dot{\theta}_s^2) \quad (19)$$

## DERIVATION OF EQUATION OF MOTION

### FOR COORDINATE $q_1$

In applying Equation (18) to the direction of the coordinate  $q_1$ , the value of  $\frac{\partial T}{\partial \dot{q}_1}$  is first computed using the kinetic energy,  $T$ , of Equation (19).

$$\frac{\partial T}{\partial \dot{q}_1} = M_1 \dot{\theta}_s \frac{\partial \dot{\theta}_s}{\partial \dot{q}_1} + M_2 \dot{\theta}_r \frac{\partial \dot{\theta}_r}{\partial \dot{q}_1}$$

The extraneous coordinates  $q_s$  and  $q_r$  may be eliminated from this expression by differentiating Equations (14) and (16) respectively. From Equation (14)

$$\dot{\theta}_s \approx \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) + \frac{M_1}{(M_1 + M_2)^2} \dot{M}_2 (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r)$$

$$\frac{\partial \dot{\theta}_s}{\partial \dot{q}_1} = 1$$

and from Equation (16)

$$\dot{\theta}_r \approx \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) + \frac{M_1}{(M_1 + M_2)^2} \dot{M}_2 (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r)$$

$$\frac{\partial \dot{\theta}_r}{\partial \dot{q}_1} = 1$$

Therefore

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_1} = & M_1 \left[ \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) + \frac{M_1}{(M_1 + M_2)^2} \dot{M}_2 (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) \right] \\ & + M_2 \left[ \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) + \frac{M_1}{(M_1 + M_2)^2} \dot{M}_2 (l_1 \dot{\theta}_s - l_2 \dot{\theta}_r) \right] \end{aligned}$$





or

$$\frac{\partial T}{\partial \dot{q}_i} = M_1 \left[ \dot{q}_i + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] + M_2 \left[ \dot{q}_i + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right]$$

The first term in the Lagrangian expression is then

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) &= M_1 \left[ \ddot{q}_i + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) - 2 \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^3} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] \\ &+ M_2 \left[ \ddot{q}_i + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) - 2 \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^3} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] \\ &+ \dot{M}_2 \left[ \dot{q}_i + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] \end{aligned}$$

$$\begin{aligned} \text{or } \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) &= (M_1 + M_2) \ddot{q}_i + \frac{M_1 \dot{M}_2}{M_1 + M_2} (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) \\ &- \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \dot{M}_2 \dot{q}_i \end{aligned} \quad (20)$$

By inspection of Equation (19), the second term of the Lagrangian expression is

$$\frac{\partial T}{\partial q_i} = 0 \quad (21)$$

The generalized force  $\mathcal{Q}_i$  is most easily computed by considering the work done by the external forces as the coordinate  $q_i$  is varied, all other coordinates remaining fixed.

$$\text{In general, } \text{Work}_i = \mathcal{Q}_i \cdot \Delta q_i \quad , \quad \text{or } \mathcal{Q}_i = \frac{\text{Work}_i}{\Delta q_i}$$

In particular, the force  $\mathcal{Q}_1$  becomes, referring to Figure 2,

$$\mathcal{Q}_1 = \frac{\text{Work}}{\Delta q_1} = \frac{(F \sin q_4 + D \cos q_4) \Delta q_1}{\Delta q_1}$$



But within the present approximations

$$\sin q_4 \approx q_4 \quad \text{and} \quad \cos q_4 \approx 1$$

$$\text{hence } Q_1 \approx F q_4 + D \quad (22)$$

Substituting the results of Equations (20), (21), and (22) into Equation (18), the equation of motion in the  $q_1$  direction becomes

$$\begin{aligned} (M_1 + M_2) \ddot{q}_1 + \frac{M_1 \dot{M}_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) - \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ + \dot{M}_2 \dot{q}_1 = F q_4 + D \end{aligned} \quad (23)$$

#### DERIVATION OF EQUATION OF MOTION

##### FOR COORDINATE $q_2$

The equation of motion in the  $q_1$  direction is obtained in an identical manner.

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_2} = M_1 \dot{q}_6 \frac{\partial \dot{q}_6}{\partial \dot{q}_2} + M_2 \dot{q}_8 \frac{\partial \dot{q}_8}{\partial \dot{q}_2}$$

and, differentiating Equation (15)

$$\dot{q}_6 = \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2)$$

$$\text{Consequently } \frac{\partial \dot{q}_6}{\partial \dot{q}_2} = 1$$

Similarly, differentiating Equation (17)



$$\ddot{\varphi} = \ddot{\varphi}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2)$$

so that

$$\frac{d\ddot{\varphi}}{d\ddot{\varphi}_2} = 1$$

Therefore

$$\frac{\partial T}{\partial \dot{\varphi}_2} = (M_1 + M_2) \left[ \dot{\varphi}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right]$$

and the first term of the Lagrangian expression becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\varphi}_2} \right) = (M_1 + M_2) \left[ \ddot{\varphi}_2 - 2 \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right] + \dot{M}_2 \left[ \dot{\varphi}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right]$$

or

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\varphi}_2} \right) = (M_1 + M_2) \ddot{\varphi}_2 - \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} (l_1 - l_2) + \dot{M}_2 \dot{\varphi}_2 \quad (24)$$

$$\text{Furthermore} \quad \frac{\partial T}{\partial \varphi_2} = 0 \quad (25)$$

The generalized force  $\mathcal{Q}_2$  is, referring to Figure 2,

$$\mathcal{Q}_2 = \frac{\text{Work}}{\Delta \varphi_2} = \frac{[-(M_1 + M_2)g + F \cos \varphi_0 - D \sin \varphi_0] \Delta \varphi_2}{\Delta \varphi_2}$$

and since

$$\sin \varphi_0 \approx \varphi_0 \quad \text{and} \quad \cos \varphi_0 \approx 1$$

$$\mathcal{Q}_2 \approx -(M_1 + M_2)g + F - D \varphi_0 \quad (26)$$



Substituting the results of Equations (24), (25), and (26) into Equation (18), the equation of motion in the  $q_2$  direction becomes

$$(M_1 + M_2) \ddot{q}_2 - \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} (l_1 \dot{q}_1) + M_2 \ddot{q}_1 = F - g(M_1 + M_2) - D \dot{q}_2 \quad (27)$$

### DERIVATION OF EQUATION OF MOTION FOR COORDINATE $q_3$

The first term of the Lagrangian expression is obtained in an identical manner to that employed for coordinates  $q_1$  and  $q_2$ . From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_3} = I_1 \dot{q}_3 + M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_3} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_3}$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_1 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_1 - l_2 \dot{q}_4)$$

so that

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_3} = \frac{M_2 l_1}{M_1 + M_2}$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_1 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_1 - l_2 \dot{q}_4)$$

so that

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_3} = -\frac{M_1}{M_1 + M_2} l_1$$





Therefore

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_3} = & I_1 \dot{q}_3 + M_1 \left[ \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] \left( \frac{M_2}{M_1 + M_2} l_1 \right) \\ & + M_2 \left[ \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \right] \left( -\frac{M_1}{M_1 + M_2} l_1 \right) \end{aligned}$$

or

$$\frac{\partial T}{\partial \dot{q}_3} = I_1 \dot{q}_3 + \frac{M_1 M_2}{M_1 + M_2} l_1 (l_1 \dot{q}_3 - l_2 \dot{q}_4)$$

and the first term of the Lagrangian expression becomes

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_3} \right) = I_1 \ddot{q}_3 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_1 (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 M_2}{M_1 + M_2} l_1 (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) \quad (28)$$

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_3} = M_1 \left( \dot{q}_1 \frac{\partial \dot{q}_3}{\partial \dot{q}_3} + \dot{q}_2 \frac{\partial \dot{q}_3}{\partial \dot{q}_3} \right) + \left( \dot{q}_1 \frac{\partial \dot{q}_3}{\partial \dot{q}_3} + \dot{q}_2 \frac{\partial \dot{q}_3}{\partial \dot{q}_3} \right) M_2 \quad (29)$$

and, differentiating Equation (14)

$$\left. \begin{aligned} \dot{q}_3 &= \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ \text{so that} \\ \frac{\partial \dot{q}_3}{\partial \dot{q}_3} &= \frac{M_1 M_2}{(M_1 + M_2)^2} l_1 \end{aligned} \right\} \quad (30)$$

Similarly, differentiating Equation (16)

$$\left. \begin{aligned} \dot{q}_4 &= \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ \text{so that} \\ \frac{\partial \dot{q}_4}{\partial \dot{q}_3} &= \frac{M_1 M_2}{(M_1 + M_2)^2} l_1 \end{aligned} \right\} \quad (31)$$



But now the values of  $\frac{\partial \dot{q}_6}{\partial \dot{q}_3}$  and  $\frac{\partial \dot{q}_8}{\partial \dot{q}_3}$  cannot be evaluated to the first order from the linearized expressions for  $q_6$  and  $q_8$  in Equations (15) and (17) respectively because the  $q_3$  term has vanished in the process of linearizing these equations. Therefore the differentiation must be carried out upon the exact expressions, Equations (10) and (8), and the results then linearized. From Equation (10)

$$q_6 = q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos \theta \quad \text{where } \theta = q_3 - \delta$$

and the time derivative is

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1 + M_2} l_4 \sin \theta \dot{\theta} + \frac{M_2}{M_1 + M_2} \dot{l}_4 \cos \theta + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_4 \cos \theta \quad (32)$$

Therefore

$$\begin{aligned} \frac{\partial \dot{q}_6}{\partial \dot{q}_3} = & -\frac{M_2}{M_1 + M_2} \frac{\partial l_4}{\partial \dot{q}_3} \sin \theta \dot{\theta} - \frac{M_2 l_4}{M_1 + M_2} \frac{\partial \sin \theta}{\partial \dot{q}_3} \dot{\theta} - \frac{M_2}{M_1 + M_2} l_4 \sin \theta \frac{\partial \dot{\theta}}{\partial \dot{q}_3} \\ & + \frac{M_2}{M_1 + M_2} \frac{\partial \dot{l}_4}{\partial \dot{q}_3} \cos \theta + \frac{M_2}{M_1 + M_2} \frac{\dot{l}_4}{\partial \dot{q}_3} \cos \theta + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{\partial l_4}{\partial \dot{q}_3} \cos \theta \\ & + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_4 \frac{\partial \cos \theta}{\partial \dot{q}_3} \end{aligned} \quad (33)$$

The value of  $\frac{\partial l_4}{\partial \dot{q}_3}$  follows from Equation (11)

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos (q_4 - q_3)}$$



and upon differentiation

$$\frac{\partial \ell_4}{\partial \theta_3} = - \frac{\ell_1 \ell_2 \sin(\theta_4 - \theta_3)}{\ell_4}$$

which, by retaining only first order terms may be written

$$\frac{\partial \ell_4}{\partial \theta_3} \approx - \frac{\ell_1 \ell_2 (\theta_4 - \theta_3)}{\ell_4}$$

Similarly

$$\frac{\partial \ell_4}{\partial \theta_4} = \dot{\ell}_4 = \frac{\ell_1 \ell_2 [\sin(\theta_4 - \theta_3)] (\dot{\theta}_4 - \dot{\theta}_3)}{\ell_4}$$

(34)

The various derivatives of  $\phi$  must be computed from

Equation (12)

$$\phi = \theta_3 - \delta = \theta_3 - \sin^{-1} \left[ \frac{\ell_2}{\ell_4} \sin(\theta_4 - \theta_3) \right]$$

Thus

$$\frac{\partial \phi}{\partial \theta_3} = 1 - \frac{\ell_2}{\sqrt{1 - \left[ \frac{\ell_2 \sin(\theta_4 - \theta_3)}{\ell_4} \right]^2}} \frac{-\ell_2 \cos(\theta_4 - \theta_3) \left[ \sin(\theta_4 - \theta_3) \right] \left( -\frac{\ell_1 \ell_2 \sin(\theta_4 - \theta_3)}{\ell_4} \right)}{\ell_4^2}$$

$$\frac{\partial \phi}{\partial \theta_3} = 1 + \frac{\ell_2}{\sqrt{\ell_4^2 - \ell_2^2 \sin^2(\theta_4 - \theta_3)}} \frac{\ell_4^2 \cos(\theta_4 - \theta_3) - \ell_1 \ell_2 \sin^2(\theta_4 - \theta_3)}{\ell_4^2}$$

and approximately

$$\frac{\partial \phi}{\partial \theta_3} \approx 1 + \frac{\ell_2}{\ell_4} \frac{\ell_4^2}{\ell_4^2} \approx \frac{\ell_1}{\ell_4} \quad (35)$$

Now

$$\frac{\partial \sin \phi}{\partial \theta_3} = \cos \phi \frac{\partial \phi}{\partial \theta_3}$$



and substituting the results of Equation (35) and linearizing

$$\frac{\partial \sin \theta}{\partial \beta_3} \approx \frac{l_1}{l_4} \cos \theta \approx \frac{l_1}{l_4} \quad (36)$$

Similarly, from the results of Equations (35) and (13)

$$\frac{\partial \cos \theta}{\partial \beta_3} = -\sin \theta \frac{\partial \theta}{\partial \beta_3} \approx -\frac{l_1 \beta_2 - l_2 \beta_4}{l_4} \frac{l_1}{l_4} = -\frac{l_1 (l_1 \beta_2 - l_2 \beta_4)}{l_4^2} \quad (37)$$

Substituting Equations (34), (35), (36), and (37) into Equation (33) and retaining only first order terms

$$\begin{aligned} \frac{\partial \dot{\beta}_2}{\partial \beta_3} = & -\frac{M_2}{M_1 + M_2} \frac{l_1 (l_1 \dot{\beta}_2 - l_2 \dot{\beta}_4)}{l_4} - \frac{M_2}{M_1 + M_2} \frac{l_1 l_2 (\dot{\beta}_4 - \dot{\beta}_2)}{l_4} \\ & - \frac{M_1 M_2}{(M_1 + M_2)^2} \frac{l_1 l_2 (\beta_4 - \beta_2)}{l_4} - \frac{M_1 M_2}{(M_1 + M_2)^2} \frac{l_1 (l_1 \beta_2 - l_2 \beta_4)}{l_4} \end{aligned}$$

or

$$\frac{\partial \dot{\beta}_2}{\partial \beta_3} = -\frac{M_2}{M_1 + M_2} \frac{l_1 \dot{\beta}_2 (l_1 - l_2)}{l_4} - \frac{M_1 M_2}{(M_1 + M_2)^2} \frac{l_1 \beta_2 (l_1 - l_2)}{l_4} \quad (38)$$

To find  $\frac{\partial \dot{\beta}_2}{\partial \beta_3}$  the same procedure is followed. Thus, from Equation (8)

$$\beta_3 = \beta_2 - \frac{M_1}{M_1 + M_2} l_4 \cos \theta \quad \text{where} \quad \theta = \beta_3 - \delta$$

and the time derivative is

$$\dot{\beta}_3 = \dot{\beta}_2 + \frac{M_1}{M_1 + M_2} l_4 (\sin \theta) \dot{\theta} - \frac{M_1}{M_1 + M_2} \dot{l}_4 \cos \theta + \frac{M_1 M_2}{(M_1 + M_2)^2} l_4 \cos \theta \quad (39)$$





Therefore

$$\begin{aligned} \frac{\partial \dot{q}_3}{\partial q_3} &= \frac{M_1}{M_1 + M_2} \frac{\partial l_4}{\partial q_3} \sin \theta \dot{\theta} + \frac{M_1}{M_1 + M_2} l_4 \frac{\partial \sin \theta}{\partial q_3} \dot{\theta} \\ &+ \frac{M_1}{M_1 + M_2} l_4 \sin \theta \frac{\partial \dot{\theta}}{\partial q_3} - \frac{M_1}{M_1 + M_2} \frac{\partial l_4}{\partial q_3} \cos \theta \\ &- \frac{M_1 \dot{q}_4 \cos \theta}{M_1 + M_2} + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{\partial l_4}{\partial q_3} \cos \theta + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_4 \frac{\partial \cos \theta}{\partial q_3} \quad (40) \end{aligned}$$

Substituting Equations (34), (35), (36), and (37) into Equation (39)

and retaining only first order terms

$$\begin{aligned} \frac{\partial \dot{q}_3}{\partial q_3} &= \frac{M_1}{M_1 + M_2} \frac{l_1}{l_4} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1}{M_1 + M_2} \frac{l_1 l_2}{l_4} (\dot{q}_4 - \dot{q}_3) \\ &- \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_1 l_2}{l_4} (q_4 - q_3) - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_1}{l_4} (l_1 q_3 - l_2 q_4) \\ \frac{\partial \dot{q}_3}{\partial q_3} &= \frac{M_1}{M_1 + M_2} \frac{l_1}{l_4} \dot{q}_3 (l_1 - l_2) - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_1}{l_4} q_3 (l_1 - l_2) \quad (41) \end{aligned}$$

The second term of the Lagrangian can now be written by

substituting Equations (30), (31), (32), (38), (39) and (41) into Equation

tion (30)

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_3} &= M_1 \left[ \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_1 \right] \\ &+ M_2 \left[ \dot{q}_2 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_1 \right] + \end{aligned}$$



$$\begin{aligned}
 &+ M_1 \left[ \dot{g}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right] \left[ -\frac{M_2}{M_1 + M_2} l_1 \dot{g}_3 - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_1 g_3 \right] \\
 &+ M_2 \left[ \dot{g}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right] \left[ \frac{M_1}{M_1 + M_2} l_1 \dot{g}_3 - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_1 g_3 \right]
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{\partial T}{\partial g_2} &= \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \dot{g}_1 + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 (l_1 g_3 - l_2 g_4) \\
 &- \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 g_3 \dot{g}_2 - \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 g_3 (l_1 - l_2) \quad (42)
 \end{aligned}$$

The generalized force  $\mathcal{Q}_3$  is

$$\mathcal{Q}_3 = \frac{\text{Work}_3}{\Delta g_3}$$

Therefore, referring to Figure 5,

$$\begin{aligned}
 \mathcal{Q}_3 &= -M_1 g \frac{\partial g_6}{\partial g_3} - M_2 g \frac{\partial g_8}{\partial g_3} + F \sin \theta_4 \frac{\partial}{\partial g_3} [g_7 - (l_2 + l_3) \sin \theta_4] \\
 &+ F \cos \theta_4 \frac{\partial}{\partial g_3} [g_8 - (l_2 + l_3) \cos \theta_4] \\
 &+ D \cos \theta_4 \frac{\partial}{\partial g_3} [g_7 - (l_2 + l_3) \sin \theta_4] \\
 &- D \sin \theta_4 \frac{\partial}{\partial g_3} [g_8 - (l_2 + l_3) \cos \theta_4]
 \end{aligned}$$







Substituting Equations (34), (36), and (37) into this expression, and using the approximations  $\sin \theta_i \approx \theta_i$  and  $\cos \theta_i \approx 1$ ,  $\mathcal{Q}_3$  becomes

$$\begin{aligned} \mathcal{Q}_3 = & (F\dot{\theta}_4 + D) \left\{ -\frac{M_1}{M_1 + M_2} \left[ l_1 + \frac{l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4}{\dot{\theta}_4} \left( -\frac{l_1 l_2 (\theta_4 - \theta_3)}{\dot{\theta}_4} \right) \right] \right\} \\ & + (F - D\dot{\theta}_4) \left\{ -\frac{M_1}{M_1 + M_2} \left[ -\frac{l_1 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4)}{\dot{\theta}_4} + \left( -\frac{l_1 l_2 (\theta_4 - \theta_3)}{\dot{\theta}_4} \right) \right] \right\} \end{aligned}$$

Finally, retaining only first order terms, the expression for the generalized force  $\mathcal{Q}_3$  reduces to

$$\mathcal{Q}_3 = -\frac{M_1 l_1}{M_1 + M_2} [F(\theta_4 - \theta_3) + D] \quad (43)$$

Therefore, by substituting the results of Equations (28), (42), and (43) into Equation (18), the equation of motion in the  $q_3$  direction becomes

$$\begin{aligned} I_3 \ddot{\theta}_3 + \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_1 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) + \frac{M_1 M_2}{M_1 + M_2} l_1 (l_1 \ddot{\theta}_3 - l_2 \ddot{\theta}_4) \\ - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \dot{\theta}_1 - \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) + \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \dot{\theta}_3 \dot{\theta}_2 \\ + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 \dot{\theta}_3 (l_1 - l_2) = -\frac{M_1 l_1}{M_1 + M_2} [F(\theta_4 - \theta_3) + D] \quad (44) \end{aligned}$$





# DERIVATION OF EQUATION OF MOTION

## FOR COORDINATE $q_4$

The first term of the Lagrangian expression is obtained in an identical manner to that employed for coordinates  $q_1$ ,  $q_2$ , and  $q_3$ .

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 + M_1 \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_4} + M_2 \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_4}$$

and, differentiating Equation (14)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4)$$

so that

$$\frac{\partial \dot{q}_5}{\partial \dot{q}_4} = -\frac{M_2 l_2}{M_1 + M_2}$$

Similarly, differentiating Equation (16)

$$\dot{q}_7 = \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4)$$

so that

$$\frac{\partial \dot{q}_7}{\partial \dot{q}_4} = \frac{M_1 l_2}{M_1 + M_2}$$

Therefore,

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_4} = & I_2 \dot{q}_4 + M_1 \left[ \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ -\frac{M_2 l_2}{M_1 + M_2} \right] \\ & + M_2 \left[ \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ \frac{M_1 l_2}{M_1 + M_2} \right] \end{aligned}$$

or

$$\frac{\partial T}{\partial \dot{q}_4} = I_2 \dot{q}_4 - \frac{M_1 M_2}{M_1 + M_2} l_2 (l_1 \dot{q}_3 - l_2 \dot{q}_4)$$



and the first term of the Lagrangian expression becomes

$$\begin{aligned} \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_4} &= I_2 \ddot{q}_4 + \dot{I}_2 \dot{q}_4 - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ &\quad - \frac{M_1 M_2}{M_1 + M_2} l_2 (l_1 \ddot{q}_3 - l_2 \ddot{q}_4) \end{aligned} \quad (45)$$

From Equation (19)

$$\frac{\partial T}{\partial \dot{q}_4} = M_1 \left( \dot{q}_5 \frac{\partial \dot{q}_5}{\partial \dot{q}_4} + \dot{q}_6 \frac{\partial \dot{q}_6}{\partial \dot{q}_4} \right) + M_2 \left( \dot{q}_7 \frac{\partial \dot{q}_7}{\partial \dot{q}_4} + \dot{q}_8 \frac{\partial \dot{q}_8}{\partial \dot{q}_4} \right) \quad (46)$$

and, differentiating Equation (14)

$$\left. \begin{aligned} \dot{q}_5 &= \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ \text{so that} \\ \frac{\partial \dot{q}_5}{\partial \dot{q}_4} &= - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 \end{aligned} \right\} \quad (47)$$

Similarly, differentiating Equation (16)

$$\left. \begin{aligned} \dot{q}_7 &= \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) \\ \text{so that} \\ \frac{\partial \dot{q}_7}{\partial \dot{q}_4} &= - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 \end{aligned} \right\} \quad (48)$$

Here again the values of  $\frac{\partial \dot{q}_6}{\partial \dot{q}_4}$  and  $\frac{\partial \dot{q}_8}{\partial \dot{q}_4}$  cannot be evaluated to the first order from the linearized expressions for  $q_c$  and  $q_s$  in Equation (15) and (17) respectively because the  $q_v$  term has vanished in the process of linearizing these equations. Therefore



the differentiation must be carried out upon the exact expressions, Equations (10) and (8), and the results then linearized.

The time derivative of  $q_6$ , defined by Equation (10), has already been found from Equation (32)

$$\dot{q}_6 = \dot{q}_2 - \frac{M_2}{M_1 + M_2} l_4 (\sin \theta) \dot{\theta} + \frac{M_2}{M_1 + M_2} \dot{l}_4 \cos \theta + \frac{M_1 M_2}{(M_1 + M_2)^2} l_4 \cos \theta \quad (32)$$

Therefore

$$\begin{aligned} \frac{\ddot{q}_6}{\partial q_4} = & -\frac{M_2}{M_1 + M_2} \frac{\partial l_4 (\sin \theta) \dot{\theta}}{\partial q_4} - \frac{M_2}{M_1 + M_2} l_4 \frac{\partial \sin \theta}{\partial q_4} \dot{\theta} \\ & - \frac{M_2}{M_1 + M_2} l_4 \sin \theta \frac{\partial \dot{\theta}}{\partial q_4} + \frac{M_2}{M_1 + M_2} \frac{\partial \dot{l}_4 \cos \theta}{\partial q_4} \\ & + \frac{M_2}{M_1 + M_2} \dot{l}_4 \frac{\partial \cos \theta}{\partial q_4} + \frac{M_1 M_2}{(M_1 + M_2)^2} \frac{\partial l_4 \cos \theta}{\partial q_4} + \frac{M_1 M_2}{(M_1 + M_2)^2} l_4 \frac{\partial \cos \theta}{\partial q_4} \quad (49) \end{aligned}$$

The value of  $\frac{\partial l_4}{\partial q_4}$  follows from Equation (11)

$$l_4 = \sqrt{l_1^2 + l_2^2 - 2 l_1 l_2 \cos (q_4 - q_3)}$$

and upon differentiation

$$\frac{\partial l_4}{\partial q_4} = \frac{l_1 l_2 \sin (q_4 - q_3)}{l_4}$$

which, by retaining only first order terms, may be written

$$\frac{\partial l_4}{\partial q_4} = \frac{l_1 l_2 (q_4 - q_3)}{l_4} \quad ]$$



Similarly

$$\frac{\partial l_4}{\partial f} = \dot{l}_4 = \frac{l_1 l_2 [\sin(\theta_4 - \theta_3)] (\dot{\theta}_4 - \dot{\theta}_3)}{l_4} \quad (50)$$

The various derivatives of  $\phi$  must be computed from

Equation (12)

$$\phi = \theta_3 - \delta = \theta_3 - \sin^{-1} \left[ \frac{l_2}{l_4} \sin(\theta_4 - \theta_3) \right]$$

Thus

$$\begin{aligned} \frac{\partial \phi}{\partial \theta_4} &= - \frac{l_2}{\sqrt{1 - \left[ \frac{l_2}{l_4} \sin(\theta_4 - \theta_3) \right]^2}} \frac{l_4 \cos(\theta_4 - \theta_3) - \left[ \sin(\theta_4 - \theta_3) \right] \frac{l_1 l_2 \sin(\theta_4 - \theta_3)}{l_4}}{l_4^2} \\ \frac{\partial \phi}{\partial \theta_4} &= - \frac{l_2}{l_4^2 - l_2^2 \sin^2(\theta_4 - \theta_3)} \frac{l_4^2 \cos(\theta_4 - \theta_3) - l_1 l_2 \sin^2(\theta_4 - \theta_3)}{l_4^2} \end{aligned}$$

and approximately

$$\frac{\partial \phi}{\partial \theta_4} \approx - \frac{l_2}{l_4} \frac{l_4^2}{l_4^2} = - \frac{l_2}{l_4} \quad (51)$$

Now

$$\frac{\partial \sin \phi}{\partial \theta_4} = \cos \phi \frac{\partial \phi}{\partial \theta_4}$$

and, substituting the results of Equation (51) and linearizing,

$$\frac{\partial \sin \phi}{\partial \theta_4} = - \frac{l_2}{l_4} \cos \phi \approx - \frac{l_2}{l_4} \quad (52)$$

Similarly, from the results of Equations (51) and (13),

$$\frac{\partial \cos \phi}{\partial \theta_4} = - \sin \phi \frac{\partial \phi}{\partial \theta_4} = \frac{l_2}{l_4} \sin \phi \approx \frac{l_2 (l_1 \theta_3 - l_2 \theta_4)}{l_4^2} \quad (53)$$





Substituting Equations (50), (51), (52), and (53) into Equation (49) and retaining only first order terms

$$\begin{aligned} \frac{\dot{f}_4}{\dot{g}_4} = & \frac{M_2}{(M_1+M_2)} \frac{l_2}{l_4} (l_1 \dot{f}_3 - l_2 \dot{f}_4) + \frac{M_2}{M_1+M_2} \frac{l_2 l_1}{l_4} (\dot{f}_4 - \dot{f}_3) \\ & + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} \frac{l_1 l_2 (q_4 - g_3)}{l_4} + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} \frac{l_2 (l_1 g_3 - l_2 g_4)}{l_4} \end{aligned}$$

or

$$\frac{\dot{q}_4}{\dot{g}_4} = \frac{M_2}{M_1+M_2} \frac{l_2}{l_4} \dot{f}_4 (l_1 - l_2) + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} \frac{l_2}{l_4} q_4 (l_1 - l_2) \quad (54)$$

To find  $\frac{\dot{q}_3}{\dot{g}_4}$  the same procedure is followed. The time derivative of  $q_3$ , defined by Equation (8), has already been found from Equation (39):

$$\dot{q}_3 = \dot{g}_2 + \frac{M_1}{M_1+M_2} l_4 (\sin \theta) \dot{\theta} - \frac{M_1}{M_1+M_2} \dot{f}_4 \cos \theta + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_4 \cos \theta \quad (39)$$

Therefore

$$\begin{aligned} \frac{\dot{q}_4}{\dot{g}_4} = & \frac{M_1}{M_1+M_2} \frac{\partial l_4}{\partial \dot{q}_4} (\sin \theta) \dot{\theta} + \frac{M_1}{M_1+M_2} l_4 \frac{\partial \sin \theta}{\partial \dot{q}_4} \dot{\theta} + \frac{M_1}{M_1+M_2} l_4 \sin \theta \frac{\partial \dot{\theta}}{\partial \dot{q}_4} \\ & - \frac{M_1}{M_1+M_2} \frac{\partial l_4}{\partial \dot{q}_4} \cos \theta - \frac{M_1}{M_1+M_2} l_4 \frac{\partial \cos \theta}{\partial \dot{q}_4} \\ & + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} \frac{\partial l_4}{\partial \dot{q}_4} \cos \theta + \frac{M_1 \dot{M}_2}{(M_1+M_2)^2} l_4 \frac{\partial \cos \theta}{\partial \dot{q}_4} \quad (55) \end{aligned}$$



Substituting Equations (50), (51), (52), and (53) into Equation (55) and retaining only first order terms

$$\begin{aligned} \frac{\partial \dot{q}_2}{\partial \dot{q}_4} = & - \frac{M_1}{M_1 + M_2} \frac{l_2}{l_4} (l_1 \dot{q}_3 - l_2 \dot{q}_4) - \frac{M_1}{M_1 + M_2} \frac{l_1 l_2}{l_4} (\dot{q}_4 - \dot{q}_3) \\ & + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_1 l_2}{l_4} (q_4 - q_3) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_2}{l_4} (l_1 q_3 - l_2 q_4) \end{aligned}$$

or

$$\frac{\partial \dot{q}_1}{\partial \dot{q}_4} = - \frac{M_1}{M_1 + M_2} \frac{l_2}{l_4} \dot{q}_4 (l_1 - l_2) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} \frac{l_2}{l_4} q_4 (l_1 - l_2) \quad (56)$$

The second term of the Lagrangian can now be written by substituting Equations (32), (39), (47), (54), and (56) into Equation (46)

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_4} = & M_1 \left[ \dot{q}_1 + \frac{M_2}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 \right] \\ & + M_2 \left[ \dot{q}_1 - \frac{M_1}{M_1 + M_2} (l_1 \dot{q}_3 - l_2 \dot{q}_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 q_3 - l_2 q_4) \right] \left[ - \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 \right] \\ & + M_1 \left[ \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right] \left[ \frac{M_2}{M_1 + M_2} l_2 \dot{q}_4 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 q_4 \right] \\ & + M_2 \left[ \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (l_1 - l_2) \right] \left[ - \frac{M_1}{M_1 + M_2} l_2 \dot{q}_4 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} l_2 q_4 \right] \end{aligned}$$

or

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_4} = & - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 \dot{q}_1 - \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^2} l_2 (l_1 q_3 - l_2 q_4) \\ & + \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 q_4 \dot{q}_2 + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^2} l_2 q_4 (l_1 - l_2) \quad (57) \end{aligned}$$



The generalized force  $\mathcal{Q}_4$  is

$$\mathcal{Q}_4 = \frac{\text{Work}_4}{dq_4}$$

Therefore, referring to Figure 5,

$$\begin{aligned}\mathcal{Q}_4 &= (F \sin q_4 + D \cos q_4) \frac{\partial}{\partial q_4} [q_1 - (l_2 + l_3) \sin q_4] \\ &\quad + (F \cos q_4 - D \sin q_4) \frac{\partial}{\partial q_4} [q_2 - (l_2 + l_3) \cos q_4] \\ &\quad - M_2 g \frac{\partial q_0}{\partial q_4} - M_1 g \frac{\partial q_c}{\partial q_4}\end{aligned}$$

Substituting Equations (8), (9), and (10) into this expression to eliminate the extraneous coordinates  $q_c$ ,  $q_1$ , and  $q_2$

$$\begin{aligned}\mathcal{Q}_4 &= (F \sin q_4 + D \cos q_4) \frac{\partial}{\partial q_4} \left[ q_1 - \frac{M_1}{M_1 + M_2} l_4 \sin \Theta - (l_2 + l_3) \sin q_4 \right] \\ &\quad + (F \cos q_4 - D \sin q_4) \frac{\partial}{\partial q_4} \left[ q_2 - \frac{M_1}{M_1 + M_2} l_4 \cos \Theta - (l_2 + l_3) \cos q_4 \right] \\ &\quad - M_2 g \frac{\partial}{\partial q_4} \left( q_3 - \frac{M_1}{M_1 + M_2} l_4 \cos \Theta \right) - M_1 g \frac{\partial}{\partial q_4} \left( q_2 + \frac{M_2}{M_1 + M_2} l_4 \cos \Theta \right)\end{aligned}$$

and collecting terms and performing the differentiation

$$\begin{aligned}\mathcal{Q}_4 &= (F \sin q_4 + D \cos q_4) \left[ -\frac{M_1}{M_1 + M_2} \left( l_4 \frac{\partial \sin \Theta}{\partial q_4} + \sin \Theta \frac{\partial l_4}{\partial q_4} \right) - (l_2 + l_3) \cos q_4 \right] \\ &\quad + (F \cos q_4 - D \sin q_4) \left[ -\frac{M_1}{M_1 + M_2} \left( l_4 \frac{\partial \cos \Theta}{\partial q_4} + \cos \Theta \frac{\partial l_4}{\partial q_4} \right) + (l_2 + l_3) \sin q_4 \right]\end{aligned}$$

Substituting Equations (13), (34), (36), and (37) into this expression,

and using the approximations  $\sin q_1 \approx q_1$  and  $\cos q_1 \approx 1$ ,  $\mathcal{Q}_4$  becomes



$$\mathcal{L}_4 \approx (F_4 + D) \left[ -\frac{M_1}{M_1 + M_2} (-l_2) - (l_2 + l_3) \right] + (F - D g_4) \left[ -\frac{M_1}{M_1 + M_2} \left( \frac{l_2 (l_1 \ddot{q}_2 - l_3 \ddot{q}_4)}{l_4} + \frac{l_1 l_2 (\ddot{q}_4 \ddot{q}_3)}{l_4} + (l_2 + l_3) \ddot{q}_4 \right) \right]$$

Finally, retaining only first order terms, the expression for the generalized force  $\mathcal{L}_4$  reduces to

$$\mathcal{L}_4 = -D \left( \frac{M_2}{M_1 + M_2} l_2 + l_3 \right) \quad (58)$$

Therefore, by substituting the results of Equations (45), (57), and (58) into Equation (18), the equation of motion in the  $q_4$  direction becomes

$$\begin{aligned} I_2 \ddot{q}_4 + I_2 \dot{q}_4 - \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_2 (l_1 \dot{q}_2 - l_3 \dot{q}_4) - \frac{M_1 M_2}{M_1 + M_2} l_2 (l_1 \ddot{q}_2 - l_3 \ddot{q}_4) \\ + \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 \dot{q}_2 + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^2} l_2 (l_1 \ddot{q}_2 - l_3 \ddot{q}_4) - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 \ddot{q}_4 \dot{q}_2 \\ - \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^2} l_2 \ddot{q}_4 (l_1 - l_2) = -D \left[ \frac{M_2}{M_1 + M_2} l_2 + l_3 \right] \end{aligned} \quad (59)$$

Thus Equations (23), (27), (44) and (59) are the four Lagrangian equations of motion corresponding to the four principal coordinates of the system.





# ANALYSIS OF THE EXTERNAL FORCES AND REDUCTION OF THE EQUATIONS TO DIMENSIONLESS FORM

In carrying out the stability analysis the state of vertical motion corresponding to the  $q_z$  direction will be considered constant during the perturbation. That is, the values of  $\dot{q}_z$  and  $\ddot{q}_z$  will be fixed, and their magnitudes will be determined by the time-instant at which the stability of the system is investigated. Therefore the equation of motion in the  $q_z$  direction, Equation (27), will not be considered further in this analysis.

The thrust force,  $F$ , of the booster may be expressed in terms of the mass rate of flow,  $\dot{M}_2$ , and the specific impulse of the fuel,  $I_{sp}$ , as

$$F = -\dot{M}_2 g I_{sp}$$

The negative sign is necessary because  $\dot{M}_2$  is numerically a negative quantity as it occurs in the Lagrangian derivation. Thus the thrust,  $F$ , will become a positive quantity when evaluated numerically in the next section.

Also, it was found in the first part of the derivation that the jet damping force,  $D$ , is proportional to the quantity  $\dot{M}_2 l_s \dot{q}_4$ . In order to make this a numerically positive quantity, it will be written as

$$D = -\dot{M}_2 l_s \dot{q}_4$$



The moment of inertia,  $I_x$ , can be written as the product of the mass,  $M_x$ , and the square of the radius of gyration,  $k_x$ , that is

$$I_x = M_x k_x^2$$

The time derivative of the moment of inertia,  $\dot{I}_x$ , is

$$\frac{dI_x}{dt} = \frac{dM_x}{dt} k_x^2 + M_x \frac{d(k_x^2)}{dt} \quad . \text{ However, the second term is}$$

small compared to the first term and will be neglected in this analysis. Therefore the following approximation can be made

$$\dot{I}_x \approx \dot{M}_x k_x^2$$

Making these four substitutions in the equations of motion in the  $q_1$ ,  $q_2$ , and  $q_4$  directions the following equations are obtained from Equations (23), (44), and (59), respectively.

From Equation (23)

$$\begin{aligned} (M_1 + M_2) \ddot{q}_1 + \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \dot{q}_3 - \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} l_1 q_3 \\ - \left( \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 - \dot{M}_2 l_1 \right) \dot{q}_4 + \left( \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^2} l_2 + \dot{M}_2 g I_{cp} \right) q_4 = 0 \end{aligned} \quad (60)$$

From Equation (44)

$$- \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \dot{q}_1 + (M_1 k_x^2 + \frac{M_1 M_2}{M_1 + M_2} l_1^2) \ddot{q}_3 + \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_1^2 \dot{q}_3 +$$



$$\begin{aligned}
 & + \left( \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 \ddot{\theta}_2 - \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 l_2 - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 g I_{sp} \right) \ddot{\theta}_3 \\
 & - \frac{M_1 M_2}{M_1 + M_2} l_1 l_2 \ddot{\theta}_4 - \left( \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 l_3 + \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_1 l_2 \right) \ddot{\theta}_4 \\
 & + \left( \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 l_2 - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_1 g I_{sp} \right) \ddot{\theta}_4 = 0
 \end{aligned} \tag{61}$$

From Equation (59)

$$\begin{aligned}
 & \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 \ddot{\theta}_1 - \frac{M_1 M_2}{M_1 + M_2} l_1 l_2 \ddot{\theta}_3 - \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_1 l_2 \ddot{\theta}_3 \\
 & + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 l_2 \ddot{\theta}_3 + \left( M_2 k_2^2 + \frac{M_1 M_2}{M_1 + M_2} l_2^2 \right) \ddot{\theta}_4 \\
 & + \left( \dot{M}_2 k_2^2 + \frac{M_1^2 \dot{M}_2}{(M_1 + M_2)^2} l_2^2 - \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 l_3 - \dot{M}_2 l_3 l_3 \right) \ddot{\theta}_4 \\
 & - \left( \frac{M_1 \dot{M}_2}{M_1 + M_2} l_2 \ddot{\theta}_2 + \frac{M_1^2 \dot{M}_2^2}{(M_1 + M_2)^3} l_1 l_2 \right) \ddot{\theta}_4 = 0
 \end{aligned} \tag{62}$$

To write these last three equations in dimensionless form introduce the dimensionless coordinates  $Q_1$ ,  $Q_2$ , and  $Q_4$ , and the characteristic units of time  $\equiv I_{sp}$ , velocity  $\equiv I_{sp} g$ , and acceleration  $\equiv g$ .

Thus

$$\left. \begin{aligned}
 Q_1 &= \frac{\theta_1}{I_{sp} g} & \dot{Q}_1 &= \frac{\dot{\theta}_1}{I_{sp} g} & \ddot{Q}_1 &= \frac{\ddot{\theta}_1}{g} \\
 Q_2 &= \theta_2 & \dot{Q}_2 &= I_{sp} \dot{\theta}_2 & \ddot{Q}_2 &= I_{sp}^2 \ddot{\theta}_2 \\
 Q_4 &= \theta_4 & \dot{Q}_4 &= I_{sp} \dot{\theta}_4 & \ddot{Q}_4 &= I_{sp}^2 \ddot{\theta}_4
 \end{aligned} \right\} \tag{63}$$



Making the above substitutions into Equation (60) and dividing the resulting equation by  $M_1 g$ , the following dimensionless equation is obtained

$$\begin{aligned}
 & \left(1 + \frac{M_2}{M_1}\right) \ddot{Q}_1 + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \dot{Q}_1 + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \frac{l_1}{l_2} \dot{Q}_3 \\
 & - \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \frac{l_1}{l_2} Q_3 - \left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \right. \\
 & \left. - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{l_1}{I_{sp} g} \right] \dot{Q}_4 + \left[ \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \right. \\
 & \left. + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \right] Q_4 = 0
 \end{aligned} \tag{64}$$

Similarly, substituting the values of Equation (63) into Equation (61) and multiplying the resulting equation by  $\frac{M_1 + M_2}{M_1^2 g l_1}$  the second dimensionless equation is obtained

$$\begin{aligned}
 & - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \dot{Q}_1 + \left[ \frac{k_2}{I_{sp}^2 g} \left(\frac{k_1}{k_2}\right)^2 \frac{l_2}{l_1} \frac{l_1}{l_2} \left(1 + \frac{M_2}{M_1}\right) + \frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \frac{l_1}{l_2} \right] \ddot{Q}_3 \\
 & + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \frac{l_1}{l_2} \dot{Q}_3 + \left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) - \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \right. \\
 & \left. + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \right] Q_3 - \frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \ddot{Q}_4 - \left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{l_1}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) \right. \\
 & \left. + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} \right] \dot{Q}_4 + \left[ \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{l_2}{k_2} - \frac{M_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \right] Q_4 = 0
 \end{aligned} \tag{65}$$





Finally, substituting the values of Equation (63) into Equation (62) and multiplying the resulting equation by  $\frac{M_1 + M_2}{M_1 M_2 g l_2}$  the third dimensionless equation is obtained

$$\begin{aligned}
 & \frac{M_1}{M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \ddot{Q}_1 - \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \frac{l_1}{l_2} \ddot{Q}_2, \\
 & - \frac{M_1}{M_2} \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \frac{l_1}{l_2} \ddot{Q}_3 + \frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{2r}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \frac{l_1}{l_2} \ddot{Q}_3 \\
 & + \left[ \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \left(1 + \frac{M_2}{M_1}\right) + \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \right] \ddot{Q}_4 + \left[ \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \frac{M_1}{M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \right. \\
 & + \frac{M_1}{M_2} \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} - \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \frac{l_2}{I_{2r} g} \left(1 + \frac{M_2}{M_1}\right) \\
 & \left. - \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{M_1}{M_2} \frac{l_2}{I_{2r} g} \left(\frac{l_2}{k_1} \frac{k_2}{l_1} - 1\right) \right] \ddot{Q}_4 \\
 & - \left[ \frac{M_1}{M_2} \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} \frac{\dot{Q}_2}{I_{2r} g} \left(1 + \frac{M_2}{M_1}\right) + \frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{2r}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2r} g} \frac{l_2}{k_1} \frac{l_1}{l_2} \right] \ddot{Q}_4 = 0 \quad (66)
 \end{aligned}$$

Equations (64), (65), and (66) are the three dimensionless equations of motion of the system and contain the following eight dimensionless parameters

$$\begin{array}{cccc}
 \frac{M_1}{M_2} & \frac{k_2}{I_{2r} g} & \frac{\dot{M}_2 I_{2r}}{M_1 + M_2} & \frac{l_1}{l_2} \\
 \frac{k_2}{k_1} & \frac{l_2}{I_{2r} g} & \frac{\dot{Q}_2}{I_{2r} g} & \frac{k_2}{l_2}
 \end{array}$$



### III. NUMERICAL INVESTIGATION OF STABILITY

Rather than attempt to find a solution for Equations (64), (65), and (66), it was decided to introduce selected values of the first six dimensionless parameters and, treating  $\frac{L_1}{L_2}$  and  $\frac{L_3}{L_2}$  as variables, to use Routh's Stability Criteria to determine the range of combinations of these two variables in which the system would be stable.

To simplify the writing of the determinantal equation necessary for the use of the Routh criteria, introduce the following notation where

$A_{ij}$  coefficient of  $Q_j$  in Equation  $j$

$B_{ij}$  coefficient of  $\dot{Q}_j$  in Equation  $j$

$C_{ij}$  coefficient of  $\ddot{Q}_j$  in Equation  $j$

and

$j = 1$  corresponds to Equation (64)

$j = 2$  corresponds to Equation (65)

$j = 3$  corresponds to Equation (66)

With this notation Equations (64), (65), and (66) may now be written as



$$\left. \begin{aligned} C_{11} \ddot{Q}_1 + B_{11} \dot{Q}_1 + B_{31} \dot{Q}_3 + A_{31} Q_3 + B_{41} \dot{Q}_4 + A_{41} Q_4 &= 0 \\ B_{12} \dot{Q}_1 + C_{32} \ddot{Q}_3 + B_{32} \dot{Q}_3 + A_{32} Q_3 + C_{42} \ddot{Q}_4 + B_{42} \dot{Q}_4 + A_{42} Q_4 &= 0 \\ B_{31} \dot{Q}_1 + C_{33} \ddot{Q}_3 + B_{33} \dot{Q}_3 + A_{33} Q_3 + C_{43} \ddot{Q}_4 + B_{43} \dot{Q}_4 + A_{43} Q_4 &= 0 \end{aligned} \right\} (67)$$

where, letting  $\frac{I_2}{k_2} \equiv \beta$  and  $\frac{I_1}{I_2} \equiv \alpha$ , the coefficients in Equation (67) are

$$A_{31} = -\left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \alpha \beta$$

$$A_{32} = \frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{\dot{g}_2}{I_{2P} g} + \frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) - \left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \beta$$

$$A_{33} = \frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \alpha \beta$$

$$A_{41} = \frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) + \left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \beta$$

$$A_{42} = -\frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) + \left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \beta$$

$$A_{43} = -\left[\frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \frac{M_1}{M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{\dot{g}_2}{I_{2P} g} + \frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{2P}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{2P}^2 g} \alpha \beta\right]$$

$$B_{11} = \frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)$$

$$B_{12} = -\frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)$$

$$B_{13} = \frac{M_1}{M_2} \frac{\dot{M}_2 I_{2P}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)$$



$$B_{31} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} d\beta$$

$$B_{32} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} d\beta$$

$$B_{33} = -\frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{M_1}{M_2} \frac{k_2}{I_{sp}^2 g} d\beta$$

$$B_{41} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{L_5}{I_{sp}^2 g} - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta$$

$$B_{42} = -\left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{L_5}{I_{sp}^2 g} + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta \right]$$

$$B_{43} = \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{M_1}{M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{k_2}{I_{sp}^2 g} \frac{1}{\beta} + \frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \beta$$

$$- \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right) \frac{L_5}{I_{sp}^2 g} - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{M_1}{M_2} \frac{L_5}{I_{sp}^2 g} \left(\frac{L_5}{k_2} \frac{1}{\beta} - 1\right)$$

$$C_{11} = \left(1 + \frac{M_2}{M_1}\right)$$

$$C_{32} = \frac{k_2}{I_{sp}^2 g} \left(\frac{k_1}{k_2}\right)^2 \left(1 + \frac{M_2}{M_1}\right) \frac{1}{d\beta} + \frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} d\beta$$

$$C_{33} = -\frac{k_2}{I_{sp}^2 g} d\beta$$

$$C_{42} = -\frac{M_2}{M_1} \frac{k_2}{I_{sp}^2 g} \beta$$

$$C_{43} = \frac{k_2}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) \frac{1}{\beta} + \frac{k_2}{I_{sp}^2 g} \beta$$





Inasmuch as oscillatory solutions are of main interest,

assume  $Q_i \propto e^{\lambda t}$

Then  $\dot{Q}_i \propto \lambda e^{\lambda t} = \lambda Q_i$

and  $\ddot{Q}_i \propto \lambda^2 Q_i$

For the system of homogeneous differential equations (67) to have other than a trivial solution, the determinant of coefficients of  $Q_i$  must vanish. That is

$$\begin{vmatrix} C_{11} \lambda^2 + B_{11} \lambda & B_{12} \lambda + A_{11} & B_{13} \lambda + A_{11} \\ B_{12} \lambda & C_{22} \lambda^2 + B_{22} \lambda + A_{22} & C_{23} \lambda^2 + B_{23} \lambda + A_{22} \\ B_{13} \lambda & C_{32} \lambda^2 + B_{32} \lambda + A_{32} & C_{33} \lambda^2 + B_{33} \lambda + A_{33} \end{vmatrix} = 0 \quad (67a)$$

Expanding the determinant of (67a)

$$\begin{aligned} & C_{11} F \lambda^6 + (C_{11} H + B_{11} F) \lambda^5 + (B_{12} H + C_{11} J - B_{12} S + B_{13} W) \lambda^4 \\ & + (C_{11} K + B_{11} J - B_{12} T + B_{13} X) \lambda^3 + (C_{11} R + B_{11} K - B_{12} U + B_{13} Y) \lambda^2 \\ & + (B_{11} R - B_{12} V + B_{13} Z) \lambda = 0 \end{aligned}$$

which can be written

$$p_0 \lambda^6 + p_1 \lambda^5 + p_2 \lambda^4 + p_3 \lambda^3 + p_4 \lambda^2 + p_5 \lambda = 0 \quad (68)$$

where

$$p_0 = C_{11} F$$

$$p_1 = C_{11} H + B_{11} F$$

$$p_2 = B_{12} H + C_{11} J - B_{12} S + B_{13} W$$

$$p_3 = C_{11} K + B_{11} J - B_{12} T + B_{13} X$$



$$p_y = C_{11} R + B_{11} K - B_{12} U + B_{13} Y$$

$$p_f = B_{11} R - B_{12} V + B_{13} Z$$

and

$$F = C_{32} C_{43} - C_{33} C_{42}$$

$$H = B_{32} C_{43} + B_{43} C_{32} - B_{42} C_{33} - B_{33} C_{42}$$

$$J = A_{32} C_{43} + A_{43} C_{32} + B_{32} B_{43} - A_{42} C_{33} - A_{33} C_{42} - B_{33} B_{42}$$

$$K = A_{43} B_{32} + A_{32} B_{43} - A_{42} B_{33} - A_{33} B_{42}$$

$$R = A_{32} A_{43} - A_{33} A_{42}$$

$$S = B_{31} C_{43} - B_{41} C_{33}$$

$$T = A_{31} C_{43} + B_{31} B_{43} - A_{41} C_{33} - B_{33} B_{41}$$

$$U = A_{43} B_{31} + A_{31} B_{43} - A_{41} B_{33} - A_{33} B_{41}$$

$$V = A_{31} A_{43} - A_{33} A_{41}$$

$$W = B_{31} C_{42} - B_{41} C_{32}$$

$$X = A_{31} C_{42} + B_{31} B_{42} - A_{41} C_{32} - B_{32} B_{41}$$

$$Y = A_{41} B_{32} + A_{42} B_{31} - A_{41} B_{32} - A_{42} B_{31}$$

$$Z = A_{31} A_{42} - A_{32} A_{41}$$

The system will be stable if the coefficients of Equation (68) and the following terms of Routh's stability criteria are all of the same sign (Reference 3)

$$\left. \begin{aligned} 1. \quad p_2 - \frac{p_2 p_3}{p_1} \\ 2. \quad p_3 - \frac{(p_1 p_4 - p_2 p_5) p_1}{p_1 p_2 - p_2 p_3} \\ 3. \quad p_4 - \frac{p_2 p_5}{p_1} - \frac{(p_1 p_2 - p_2 p_3)^2 p_1}{p_1 [p_3 (p_1 p_2 - p_2 p_3) - (p_1 p_4 - p_2 p_5)]} \end{aligned} \right\} \quad (69)$$



# NUMERICAL EXAMPLE FOR "V-2" TYPE ROCKET

To find a numerical solution to the problem the "V-2" rocket (weight approximately 28,000 lbs.) was chosen as the main rocket,  $M_1$ , and a small five-second booster providing an initial acceleration of 1-1/2 g was selected as the booster,  $M_2$ .

The dimensionless parameters of the system were then estimated to be

$$\begin{aligned} \frac{M_1}{M_2} &= 7.75 & \frac{k_1}{I_{sp}^2 g} &= 1.9 \times 10^{-6} & \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} &= -2.21 \\ \frac{k_1}{k_2} &= 7.75 & \frac{k_5}{I_{sp}^2 g} &= 2.48 \times 10^{-6} \\ \frac{\dot{q}_2}{I_{sp} g} &= 0 \quad (\text{at launching}) \end{aligned}$$

The stability of the system was then to be investigated for various values of the two variables  $\alpha = \frac{k_1}{k_2}$  and  $\beta = \frac{k_5}{k_2}$ .

Because of the inherent complexity of the constants of Equation (69), a vast amount of time would be required to investigate every likely combination of the variables  $\alpha$  and  $\beta$ . However, the writer optimistically set out to evaluate these constants for various combinations of  $\alpha$  and  $\beta$ . Inasmuch as no combination was found where even the constants  $p_1$  to  $p_7$  were of the same sign, much less where the test functions of Equation (69) were of the same sign, it was decided to reduce the complexity of the system in order to expedite the computation.



Accordingly,  $\dot{Q}_1$  and  $\ddot{Q}_1$  were assumed to be zero, that is, the system was restricted to vertical and rotational movement of the center of gravity. This assumption reduced the system of equations to two, namely, Equations (65) and (66), and reduced the labor involved in applying the Routh stability criteria substantially. Once a stable range of  $\alpha$  and  $\beta$  had been determined, these values were then to have been used in the general case for closer investigation.

#### STABILITY OF SYSTEM WITH TWO DEGREES OF FREEDOM

If both  $\dot{Q}_1$  and  $\ddot{Q}_1$  are assumed to be zero, Equations (65) and (66) may be re-written (using the abbreviated notation introduced earlier) as

$$C_{32}\ddot{Q}_3 + B_{32}\dot{Q}_3 + A_{32}Q_3 + C_{42}\ddot{Q}_4 + B_{42}\dot{Q}_4 + A_{42}Q_4 = 0$$

$$C_{33}\ddot{Q}_3 + B_{33}\dot{Q}_3 + A_{33}Q_3 + C_{43}\ddot{Q}_4 + B_{43}\dot{Q}_4 + A_{43}Q_4 = 0$$

and assuming  $Q_i = e^{\lambda t}$ , as in the more general case, the determinant for this case becomes

$$\begin{vmatrix} C_{32}\lambda^2 + B_{32}\lambda + A_{32} & C_{42}\lambda^2 + B_{42}\lambda + A_{42} \\ C_{33}\lambda^2 + B_{33}\lambda + A_{33} & C_{43}\lambda^2 + B_{43}\lambda + A_{43} \end{vmatrix} = 0$$

The expanded determinant may then be written





$$\begin{aligned} & (C_{32}C_{43} - C_{42}C_{33})\lambda^4 + (C_{32}B_{43} + B_{32}C_{43} - C_{42}B_{33} - B_{42}C_{33})\lambda^3 \\ & + (C_{32}A_{43} + A_{32}C_{43} + B_{32}B_{43} - C_{42}A_{33} - A_{42}C_{33} - B_{42}B_{33})\lambda^2 \\ & + (B_{32}A_{43} + A_{32}B_{43} - B_{42}A_{33} - A_{42}B_{33})\lambda + (A_{32}A_{43} - A_{42}A_{33}) = 0 \quad (70) \end{aligned}$$

Furthermore, since the parameter  $\frac{\dot{g}_2}{I_{sp}g}$  equals zero at launching, the last term of Equation (70) becomes

$$\begin{aligned} A_{32}A_{43} - A_{42}A_{33} = & -\left[\frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)\right] \left[\frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \Delta\beta\right] \\ & + \left[\frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)\right] \left[\frac{M_1}{M_2} \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \frac{k_2}{I_{sp}^2 g} \Delta\beta\right] = 0 \end{aligned}$$

Therefore Equation (70) reduces to a third order equation of the form

$$\rho_0 \lambda^3 + \rho_1 \lambda^2 + \rho_2 \lambda + \rho_3 = 0 \quad (71)$$

where

$$\begin{aligned} \rho_0 &= C_{32}C_{43} - C_{42}C_{33} \\ \rho_1 &= C_{32}B_{43} + B_{32}C_{43} - C_{42}B_{33} - B_{42}C_{33} \\ \rho_2 &= C_{32}A_{43} + A_{32}C_{43} + B_{32}B_{43} - C_{42}A_{33} - A_{42}C_{33} - B_{42}B_{33} \\ \rho_3 &= B_{32}A_{43} + A_{32}B_{43} - B_{42}A_{33} - A_{42}B_{33} \end{aligned}$$

and the Routh stability criteria require that  $\rho_0$ ,  $\rho_1$ ,  $\rho_2$ ,  $\rho_3$ , and  $\left(\rho_2 - \frac{\rho_1 \rho_3}{\rho_0}\right)$  all be of the same sign. (Reference 3).

The coefficients of Equation (71) may now be evaluated by substituting the parameters previously selected for the "V-2" rocket with five second booster. Collecting terms these coefficients may



be expressed as follows

$$P_2 = \frac{1}{\alpha} (a_0 d^2 + c_0) \quad (72)$$

where

$$a_0 = 5.24 \cdot 10^{-13}$$

$$c_0 = \frac{2.25 \cdot 10^{-10}}{\beta^2} + 2.45 \cdot 10^{-10}$$

$$P_1 = \frac{1}{\alpha} (a_1 d^2 + c_1) \quad (73)$$

where

$$a_1 = -1.91 \cdot 10^{-12}$$

$$c_1 = \frac{3.71 \cdot 10^{-9}}{\beta^2} - \frac{6.15 \cdot 10^{-9}}{\beta} - 4.2 \cdot 10^{-9}$$

$$P_2 = a_2 d^2 + b_2 d + c_2 \quad (74)$$

where

$$a_2 = -1.76 \cdot 10^{-11} \beta^2$$

$$b_2 = 3.53 \cdot 10^{-11} \beta^2 + 4.73 \cdot 10^{-6} \beta - 1.21 \cdot 10^{-10}$$

$$c_2 = -\left(1.76 \cdot 10^{-11} \beta^2 + 4.73 \cdot 10^{-6} \beta + \frac{5.33 \cdot 10^{-6}}{\beta} - 9.3 \cdot 10^{-9}\right)$$

$$P_3 = a_3 d^2 + b_3 d + c_3 \quad (75)$$

where

$$a_3 = 3.02 \cdot 10^{-10} \beta^2$$

$$b_3 = -(6.06 \cdot 10^{-10} \beta^2 + 8.12 \cdot 10^{-5} \beta)$$

$$c_3 = 2.03 \cdot 10^{-10} \beta^2 + 8.12 \cdot 10^{-5} \beta + 1.19 \cdot 10^{-9} - \frac{2.18 \cdot 10^{-5}}{\beta}$$



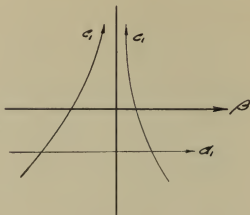
Before computing numerical values of the coefficients of Equation (71) it will be profitable to draw certain conclusions from an inspection of Equations (72) through (75).

$\alpha$  is defined as the quantity  $\frac{\ell}{k_1}$ , and  $\beta$  is defined as the quantity  $\frac{\ell_2}{k_2}$ , where  $\ell$  and  $k_1$  are inherently positive quantities by the physical nature of the problem. Therefore, it can be concluded that for any real solution of the physical problem, both  $\alpha$  and  $\beta$  must be of the same sign, that is, either both positive, or both negative.

In Equation (72) the sign of  $p_0$  is independent of  $\beta$  and takes the sign of  $\alpha$ . Therefore numerical values of  $p_0$  need not be determined at this stage of the investigation since the sign of  $p_0$  will be evident by inspection.

In Equation (73)  $a_1$  is always negative, and  $c_1$  has roots at  $\beta = 0.456$  and  $\beta = -1.93$ ,

Since both  $a_1$  and  $c_1$  are negative for  $\beta < 1.93$  and since  $\alpha$  is negative when  $\beta$  is negative, it follows that  $p_1$  is always positive in this region.



Negative  $\alpha$  yields negative  $p_0$  from Equation (72). Therefore  $p_1$  and  $p_2$  will be of opposite sign for all values of  $\beta < 1.93$ , and,



by Routh's stability criteria, stability is impossible for this case.

By analogous reasoning stability is impossible in the range

$$0 < \beta < 0.456.$$

In Equation (74)  $a_2 = 0$  when  $\beta = 0$ , and  $a_2$  is negative for all other values of  $\beta$ . Furthermore,  $b_2$  has a root at  $\beta = 2.54 \times 10^{-4}$ ,

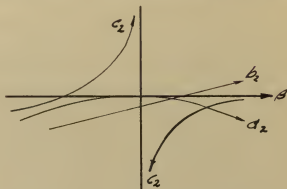
and  $c_2$  has a root at

$$\beta = -2.68 \times 10^5.$$

$a_2$ ,  $b_2$ , and  $c_2$  are all

negative in the range

$$0 < \beta < 2.54 \cdot 10^{-4}$$



it can be concluded that stability is impossible in this range by reasoning analogous to that in the case of  $p_1$ .

In Equation (75)  $a_3 = 0$  when  $\beta = 0$ , and  $a_3$  is positive for all other values of  $\beta$ .

Furthermore  $b_3$  has a

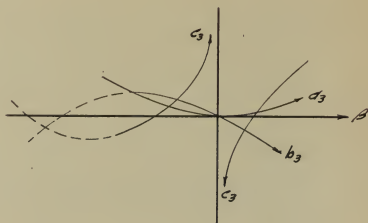
root at  $\beta = -1.34 \times 10^5$

and at  $\beta = 0$ . Also

$c_3$  has roots at  $\beta =$

$$0.456, \beta = -1.92,$$

$$\text{and } \beta = -2.68 \times 10^5.$$



No conclusions can be

drawn in this case.

To summarize the conclusions of the preceding paragraphs,





the possibility of stability need only be investigated in the ranges  $-1.93 < \beta < 0$  and  $2.54 \times 10^{-4} < \beta < 0.456$ .

In evaluating the coefficients  $p_1$ ,  $p_2$ , and  $p_3$ , the following procedure was adopted. Selected values of  $\beta$  within the positive range  $2.54 \times 10^{-4} < \beta < 0.456$  were used to find the constant coefficients of Equations (73), (74), and (75). Then the positive roots of the resulting equations were computed as functions of  $\alpha$ . Negative roots were not investigated since  $\alpha$  must be of the same sign as  $\beta$ . These values are tabulated in Table I.

Similarly, values of the coefficients and corresponding negative roots of Equations (73), (74), and (75) were computed and are tabulated in Table II.

The data of Tables I and Table II are plotted qualitatively in Figure 6 and Figure 7.

In Figure 8 and Figure 9 are plotted the logarithms of maximum and minimum limits of  $\alpha$  for which  $p_1$ ,  $p_2$ , and  $p_3$  are each positive.

It is apparent from Figure 8 that there is no single value of  $\alpha$  for which  $p_1$ ,  $p_2$ , and  $p_3$  can all be positive since  $\alpha_{min}$  for positive  $p_3$  greatly exceeds  $\alpha_{min}$  for positive  $p_1$  for all values of  $2.54 \times 10^{-4} < \beta < 0.456$ .

Likewise, in Figure 9 there is no single value of  $\alpha$  for



which  $p_1$ ,  $p_2$ , and  $p_3$  can all be negative since  $|a_{m..}|$  for negative  $p_1$  greatly exceeds  $|a_{m..}|$  for negative  $p_2$  for all values of  $-1.93 < \beta < 0$ .

Therefore, it is clear that Equations (65) and (66) do not have a stable solution for any combination of  $\alpha$  and  $\beta$  for the parameters chosen.

### STABILITY OF SYSTEM WITH REDUCED MASS RATIO

In view of the negative results obtained in the previous case it was decided to investigate the effect of changing the parameter  $\frac{M_1}{M_2}$ . The following set of dimensionless parameters was then selected

$$\frac{M_1}{M_2} = 1.5 \quad \frac{k_2}{I_{y^2} g} = 6.72 \cdot 10^{-6} \quad \frac{M_2 I_{y^2}}{M_1 + M_2} = -2.21$$

$$\frac{k_1}{k_2} = 1.5 \quad \frac{l_5}{I_{y^2} g} = 8.75 \cdot 10^{-6}$$

$$\frac{\dot{\beta}_0}{I_{y^2} g} = 0 \quad (\text{at launching})$$

Note that  $\frac{M_1}{M_2}$  was decreased from 7.75 to 1.5,  $\frac{k_2}{I_{y^2} g}$  was increased from  $1.96 \times 10^{-6}$  to  $6.72 \times 10^{-6}$ , and  $\frac{l_5}{I_{y^2} g}$  was increased from  $2.48 \times 10^{-6}$  to  $8.75 \times 10^{-6}$ . The remaining parameters



were unchanged. Thus, this second set of parameters describe a system essentially the same as that first investigated except that the mass ratio  $\frac{M_1}{M_2}$  has been markedly reduced.

The coefficients of Equation (71) are again evaluated by substituting these new parameters. Collecting terms the coefficients may be expressed as follows

$$P_0 = \frac{1}{\alpha^2} (a_0 \alpha^2 + c_0) \quad (76)$$

$$\text{where } a_0 = 5.04 \cdot 10^{-11}$$

$$c_0 = \frac{2.82 \cdot 10^{-10}}{\beta^2} + 1.70 \cdot 10^{-10}$$

$$P_1 = \frac{1}{\alpha} (a_1 \alpha^2 + c_1) \quad (77)$$

$$\text{where } a_1 = 2.49 \cdot 10^{-11}$$

$$c_1 = \frac{1.08 \cdot 10^{-9}}{\beta^2} - \frac{1.21 \cdot 10^{-9}}{\beta} - 5.62 \cdot 10^{-10}$$

$$P_2 = a_2 \alpha^4 + b_2 \alpha^2 + c_2 \quad (78)$$

$$\text{where } a_2 = -2.22 \cdot 10^{-10} \beta^2$$

$$b_2 = 4.43 \cdot 10^{-10} \beta^2 + 2.48 \cdot 10^{-5} \beta - 6.36 \cdot 10^{-10}$$

$$c_2 = -\left(2.21 \cdot 10^{-10} \beta^4 + 2.48 \cdot 10^{-5} \beta^2 + \frac{4.13 \cdot 10^{-5}}{\beta} + 1.61 \cdot 10^{-7}\right)$$

$$P_3 = a_3 \alpha^2 + b_3 \alpha + c_3 \quad (79)$$

$$\text{where } a_3 = 7.32 \cdot 10^{-10} \beta^2$$

$$b_3 = -(1.46 \cdot 10^{-9} \beta^2 + 8.22 \cdot 10^{-5} \beta)$$

$$c_3 = 7.32 \cdot 10^{-10} \beta^2 + 8.22 \cdot 10^{-5} \beta - \frac{1.57 \cdot 10^{-4}}{\beta} + 1.77 \cdot 10^{-4}$$



As previously stated,  $a$  and  $\beta$  must be of the same sign.

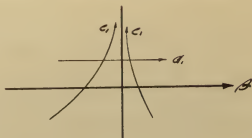
Again, the sign of  $p_c$  is independent of  $\beta$  and takes the sign of  $a$ .

In Equation (77)  $a_1$

is always positive. Also

$c_1$  has roots at  $\beta = 0.68$

and at  $\beta = -2.84$ .



In Equation (78)

$a_2 = 0$  when  $\beta = 0$ , and

$a_2$  is negative for all

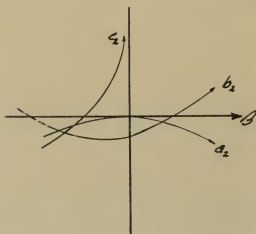
other values of  $\beta$ .

Furthermore  $b_2$  has roots

at  $\beta = -5.6 \times 10^{-5}$  and at

$\beta = 2.55 \times 10^{-5}$ . Also  $c_2$

has a root at  $\beta = -1.12 \times 10^{-5}$ .



Since  $a_1$ ,  $b_1$ , and  $c_1$  are all negative in the range  $0 < \beta < 2.55 \times 10^{-5}$  stability is not possible in this range.

In Equation (79)  $a_3 = 0$  when  $\beta = 0$ , and  $a_3$  is positive for

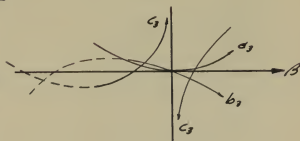
all other values of  $\beta$ . Furth-

ermore  $b_3$  has roots at  $\beta = 0$

and at  $\beta = -5.62 \times 10^{-5}$ . Also,

$c_3$  has roots at  $\beta = 0.625$ ,

$\beta = -2.83$  and  $\beta = -1.12 \times 10^{-5}$ .







It may be concluded that the possibility of stability need only be investigated outside the range  $0 < \beta < 2.55 \times 10^{-5}$ .

As in the previous case selected values of both positive and negative  $\beta$  were substituted into Equations (77), (78), and (79), to compute the constant coefficients of Equation (71), and the resulting equations were then solved for maximum and minimum allowable values of  $\alpha$ . The results are tabulated in Table III and Table IV, and are plotted qualitatively in Figure 10 and Figure 11.

Figure 12 and Figure 13 are again logarithmic plots of the maximum and minimum limits of  $\alpha$  versus the logarithm of  $\beta$ .

In Figure 12 it can be seen that stability is possible in the range  $10^{-5} < \beta < 4.5 \times 10^{-4}$  only if  $\alpha_{max}$  for positive  $p_z$  is greater than  $\alpha_{min}$  for positive  $p_y$ . (The curves appear to coincide.) However, an analysis of Table III reveals that for all values of  $10^{-5} < \beta < 0.68$  both  $c_z$  and  $c_y$  are negative whereas  $a_z$  is negative while  $a_y$  is positive in this range. Therefore

$$\alpha_{z,max} = \left| \frac{b_1}{2c_1} \right| \left[ 1 + \sqrt{1 - \left| \frac{4c_2 c_3}{b_2^2} \right|} \right] \quad \text{while} \quad \alpha_{y,min} = \left| \frac{b_1}{2c_1} \right| \left[ 1 + \sqrt{1 + \left| \frac{4c_2 c_3}{b_3^2} \right|} \right]$$

Thus it follows that  $\alpha_{y,min}$  is greater than  $\alpha_{z,max}$  in this range and stability is impossible.

Similarly, in Figure 13 stability is only possible if

$|\alpha_{max}|$  for negative  $p_y$  is greater than  $|\alpha_{min}|$  for negative  $p_z$ .



However, an analysis of Table IV reveals that for all values of  $-2.8 < \beta < -10^{-5}$  both  $c_1$  and  $c_2$  are positive while  $a_1$  is negative and  $a_2$  is positive in this range. Therefore,  $|\alpha_{1,m,1}| = \left| \frac{d_1}{2d_2} \right| \left[ 1 + \sqrt{1 + \left| \frac{4d_2 c_1}{d_1^2} \right|} \right]$  while  $|\alpha'_{1,m,1}| = \left| \frac{d_1}{2d_2} \right| \left[ 1 + \sqrt{1 - \left| \frac{4d_2 c_1}{d_1^2} \right|} \right]$  and thus  $|\alpha_{1,m,1}|$  is greater than  $|\alpha'_{1,m,1}|$ .

Therefore it is only necessary to investigate more closely the range  $0.68 < \beta < 4.5 \times 10^{-4}$  and  $-10^3 < \beta < -2.8$ .

Returning to Equation (71) and multiplying the analytic expressions for the coefficients of  $p_1$  and  $p_2$  the following equations are derived if only first order terms are retained.

$$\begin{aligned} P_2 &= a_2 d^2 + b_2 d + c_2 \\ &= - \left[ \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp} g} \right)^2 \beta^2 \right] d^2 \\ &\quad + \left[ 2 \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp} g} \right)^2 \beta^2 - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp} g} \left( 1 + \frac{M_2}{M_1} \right) \beta \right] d \\ &\quad + \left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp} g} \left( 1 + \frac{M_2}{M_1} \right) \beta + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp} g} \left( 1 + \frac{M_2}{M_1} \right)^2 \frac{1}{\beta} \right. \\ &\quad \left. - \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp} g} \right)^2 \left( 1 + \frac{M_2}{M_1} \right) \left( \frac{M_1}{M_2} \frac{R_1^2}{k_1^2} + 1 \right) \right] \end{aligned} \quad (80)$$

and

$$\begin{aligned} P_3 &= a_3 d^2 + b_3 d + c_3 \\ &= - \left[ \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^3 \left( \frac{k_2}{I_{sp} g} \right)^2 \frac{M_1}{M_2} \beta^2 \right] d^2 + \end{aligned}$$



$$\begin{aligned}
 & + \left[ 2 \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp}^2 g} \right)^2 \frac{M_1}{M_2} \beta^2 - \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right) \frac{M_1}{M_2} \beta \right] \alpha \\
 & + \left[ \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right) \frac{M_1}{M_2} \beta + \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right)^3 \frac{M_1}{M_2} \left( 1 - \frac{l_s^2}{k_2^2} \right) \frac{1}{\beta} \right. \\
 & \left. + \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( 1 + \frac{M_2}{M_1} \right)^2 \frac{M_1}{M_2} \frac{l_s}{I_{sp}^2 g} \right] \quad (81)
 \end{aligned}$$

Factoring  $\frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}$  from Equation (81)

$$\begin{aligned}
 \beta = & \frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left\{ - \left[ \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp}^2 g} \right)^2 \beta^2 \right] \alpha^2 \right. \\
 & + \left[ 2 \left( \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \right)^2 \left( \frac{k_2}{I_{sp}^2 g} \right)^2 \beta^2 - \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right) \beta \right] \alpha \\
 & + \left[ \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right) \beta + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left( 1 + \frac{M_2}{M_1} \right)^3 \left( 1 - \frac{l_s^2}{k_2^2} \right) \frac{1}{\beta} \right. \\
 & \left. \left. + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left( 1 + \frac{M_2}{M_1} \right)^2 \frac{l_s}{I_{sp}^2 g} \right] \right\} \quad (82)
 \end{aligned}$$

Comparing Equation (80) with Equation (82) it can be seen that both  $a_1$  and  $b_1$  are multiples of  $a_2$  and  $b_2$  respectively by the factor  $\frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}$ . For positive  $\beta$ , stability exists only if  $\alpha_{2, \max}$  is greater than  $\alpha_{2, \min}$ , that is, if  $C_2 < \frac{C_3}{\frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}}$



But the factor  $\frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}$  is numerically negative so that the condition for stability can be written

$$c_2 > \frac{c_3}{\frac{M_1}{M_2} \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}}$$

Substituting the values of  $c_2$  and  $c_3$  from Equation (80) and Equation (82) this condition becomes

$$\begin{aligned} & \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) \frac{1}{\beta} - \left(\frac{\dot{M}_2 I_{sp}}{M_1 + M_2}\right)^2 \left(\frac{k_2}{I_{sp}^2 g}\right)^2 \left(1 + \frac{M_2}{M_1}\right) \left(\frac{M_1}{M_2} \frac{k_1^2}{k_2^2} + 1\right) \\ & > \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \frac{k_2}{I_{sp}^2 g} \left(1 + \frac{M_2}{M_1}\right) \left(1 - \frac{\beta^2}{k_2^2}\right) \frac{1}{\beta} + \frac{\dot{M}_2 I_{sp}}{M_1 + M_2} \left(1 + \frac{M_2}{M_1}\right)^2 \frac{\beta}{I_{sp}^2 g} \end{aligned}$$

which, upon collecting terms, becomes

$$\beta > \frac{\left(1 + \frac{M_2}{M_1}\right) \left[1 - \left(1 + \frac{M_2}{M_1}\right) \left(1 - \frac{\beta^2}{k_2^2}\right)\right]}{\left(1 + \frac{M_2}{M_1}\right) \frac{\beta}{k_2} + \frac{k_2}{I_{sp}^2 g} \left(\frac{M_1}{M_2} + 1\right) \frac{\dot{M}_2 I_{sp}}{M_1 + M_2}}$$

Inserting the numerical values of the parameters, the second term of the denominator is of the order  $10^{-6}$  since  $\frac{k_2}{I_{sp}^2 g} = 6.72 \cdot 10^{-6}$ .

Therefore this term can be neglected in comparison with the first term of the denominator and the expression simplifies to

$$\beta > \frac{1 - \left(1 + \frac{M_2}{M_1}\right) \left(1 - \frac{\beta^2}{k_2^2}\right)}{\frac{\beta}{k_2}} \quad (83)$$

Solving this inequality with the parameters chosen, the result is

$$\beta > 1.62$$





As an example, choose  $\beta = 2$ . Inserting values from Table III into Equation (78) and Equation (79), for  $\beta = 2$  the following values of  $\alpha$  are obtained

$$\alpha_1 = -\frac{b}{2a} \left[ 1 \pm \sqrt{1 - \frac{4ac}{b^2}} \right] \approx -\frac{b}{2a} \left[ 1 \pm \left( 1 - \frac{2ac}{b^2} \right) \right]$$

$$\alpha \approx -\frac{b}{a} \left[ 1 - \frac{ac}{b^2} \right]$$

$$\alpha_2 \approx 5.6 \cdot 10^{-5} (1 - 2.54 \cdot 10^{-5})$$

$$\alpha_3 \approx 5.6 \cdot 10^{-5} (1 - 9.85 \cdot 10^{-5})$$

Thus  $p_z$  changes from positive to negative when  $\alpha_1 = 5.6 \times 10^{-5} (1 - 2.54 \times 10^{-5})$ , while  $p_y$  changes from negative to positive when  $\alpha_2 = 5.6 \times 10^{-5} (1 - 9.85 \times 10^{-5})$ , and both  $p_z$  and  $p_y$  are positive in the range  $(5.6 \times 10^{-5} - 5.52) < \alpha < (5.6 \times 10^{-5} - 1.42)$ . Results with a similar sensitivity are obtained for larger values of  $\beta$  but smaller values of  $\alpha$ . (It can be seen from Figure 12 that the product of  $\alpha$  and  $\beta$  is approximately constant and equal to  $10^{-5}$ .)

There is yet to be applied the final Routh criterion, namely that  $\rho_2 - \frac{\rho_2 \rho_3}{\rho_1}$  be positive also. Choosing  $p_y$  at its smallest allowable value, say  $0_+$ , and calling the corresponding value of  $p_z$  some small positive value, say  $\epsilon$ ,  $\left( \rho_2 - \frac{\rho_2 \rho_3}{\rho_1} \right) = \epsilon$ . Thus all the stability criteria are satisfied and the system is stable. Therefore, it can be concluded that the system is stable



for  $\beta > 1.62$  but that, for any given value of  $\beta$  in this stable range,  $\alpha$  is extremely restricted.

Finally, there remains the negative range of  $\beta$  to be investigated, namely  $-10^2 < \beta < -2.8$ . But, for negative values of  $\beta$ ,  $\beta$  must be greater than  $-1.62$  by Equation (83) in order that  $p_2$  and  $p_3$  both be negative. Therefore stability is not possible for negative values of  $\beta$ .



#### IV. DISCUSSION OF RESULTS

Only two possible sets of parameters for a launching system of this type have been investigated. In the first case where the mass ratio was 7.75 it was found that stability was impossible, whereas, in the second case where the mass ratio was 1.5 it was found that stability was theoretically possible for values of  $\beta$  between  $1.62$  and  $4.5 \times 10^{-4}$  if the effect of lateral motion of the center of gravity of the system was neglected. For any given value of  $\beta$  in this range stability was possible for only one corresponding value of  $\alpha$ .

Returning to the physical aspects of the problem consider now the meaning of the variables  $\alpha$  and  $\beta$ .  $\beta$  is defined as  $\frac{l_2}{k_2}$  where  $l_2$  is the distance from the center of gravity of the booster,  $M_2$ , to the pin connecting it to the strut to the main rocket,  $M_1$ , and  $k_2$  is the radius of gyration of the booster,  $M_2$ . For any system of this type  $k_2$  will always be of the order of one foot or greater. Therefore,  $l_2 = \beta k_2$  will always be equal to or greater than  $\beta$ . That is, as a first approximation,  $l_2$  is approximately numerically equal to  $\beta$ . Thus, from a practical consideration  $l_2$ , and therefore  $\beta$ , must certainly not exceed some small number of the order of 1 to 10, depending on the size of the booster, for otherwise the pin will be located too near the booster exhaust.



Now  $\alpha$  is defined as  $\frac{L}{L_2}$  where  $L$  is the length of the strut from the pin in the booster  $M_2$  to the center of gravity of the main rocket,  $M_1$ . Here again there is a practical upper limit for  $L$ , first, because the strut was assumed to be of zero mass in the derivation of the equations of motion, and second, because the strut was assumed to be of infinite rigidity. A long strut would present structural problems in rigidity since it is essentially a column acting under both axial and bending loads. Restricting  $L_2$  to values of the order of a few feet it is clear that  $\alpha$  should not exceed 100 or 200 at most, since these values correspond to an  $L_2$  of 25 to 100 feet.

In view of the foregoing remarks consider the product of the two variables  $\alpha$  and  $\beta$ . Thus  $\alpha\beta = \frac{L_1}{L_2} \frac{L_2}{k_2} = \frac{L_1}{k_2}$ . If  $k_2$  is of the order of 1 to 10 feet, say, and  $L_1$  is restricted to 25 to 100 feet, then it can be seen that the product  $\alpha\beta$  should not exceed 25. Returning to Figure 12 note that, in the range  $1.62 < \beta < 4.54 \times 10^{-4}$ ,  $\alpha\beta$  is approximately constant and equal to about  $10^{-5}$ . Thus, even though theoretically stable, this system is of no practical significance for the mass ratio investigated.





## V. CONCLUSIONS

1. A compound pendulum system of launching stabilization is not stable for a mass ratio 7.75.

2. Such a system is theoretically stable for a mass ratio 1.5 if the effect of lateral motion is neglected. However, even in this restricted case the system is of no practical significance due to the excessive length requirements for the strut from the main rocket to the booster.

3. Since the system is quite sensitive to the particular parameters chosen, several more investigations of specific cases are required before any general conclusions can be drawn as to the practicality of this method of stabilization.



TABLE I  
Data from Equations (73), (74), and (75).  
( $2.54 \times 10^{-4} < \rho < .455$ )

$\rho$	$2.54 \times 10^{-4}$	$10^{-3}$	$3 \times 10^{-3}$	0.01	0.04	0.05	0.1	0.2	0.3	0.4	0.455
$\alpha_1$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$	$-1.91 \times 10^{-4}$
$\alpha_2$	$5.76 \times 10^{-4}$	$3.71 \times 10^{-3}$	$4.1 \times 10^{-3}$	$3.65 \times 10^{-3}$	$2.17 \times 10^{-3}$	$1.38 \times 10^{-3}$	$3.05 \times 10^{-3}$	$5.9 \times 10^{-3}$	$1.68 \times 10^{-2}$	$4.2 \times 10^{-2}$	$2.1 \times 10^{-1}$
$\alpha_{31}$	$1.73 \times 10^{-4}$	$4.42 \times 10^{-4}$	$1.46 \times 10^{-3}$	$4.42 \times 10^{-4}$	$1.06 \times 10^{-3}$	850	400	172	86.4	47	10
$\alpha_{32}$	0	0	0	0	0	0	0	0	0	0	0
$\alpha_4$	$-1.13 \times 10^{-4}$	$-1.76 \times 10^{-4}$	$-1.58 \times 10^{-4}$	$-1.76 \times 10^{-4}$	$-2.82 \times 10^{-4}$	$-4.41 \times 10^{-4}$	$-1.76 \times 10^{-3}$	$-7.05 \times 10^{-3}$	$-1.59 \times 10^{-2}$	$-2.82 \times 10^{-2}$	$-3.67 \times 10^{-2}$
$\alpha_5$	$1.08 \times 10^{-3}$	$4.63 \times 10^{-3}$	$1.44 \times 10^{-2}$	$4.74 \times 10^{-3}$	$1.90 \times 10^{-2}$	$2.38 \times 10^{-2}$	$4.76 \times 10^{-2}$	$9.52 \times 10^{-2}$	$1.32 \times 10^{-1}$	$1.9 \times 10^{-1}$	$2.16 \times 10^{-1}$
$\alpha_6$	$-2.1 \times 10^{-2}$	$-5.33 \times 10^{-3}$	$-1.78 \times 10^{-2}$	$-5.33 \times 10^{-3}$	$-1.33 \times 10^{-2}$	$-1.06 \times 10^{-2}$	$-5.38 \times 10^{-2}$	$-2.75 \times 10^{-1}$	$-1.9 \times 10^{-1}$	$-1.51 \times 10^{-1}$	$-1.39 \times 10^{-1}$
$\alpha_{41}$	$9.37 \times 10^{-4}$	$2.7 \times 10^{-3}$	$8.92 \times 10^{-3}$	$2.69 \times 10^{-3}$	$6.76 \times 10^{-3}$	$5.4 \times 10^{-3}$	$2.7 \times 10^{-2}$	$1.35 \times 10^{-1}$	$8.4 \times 10^{-1}$	$6.74 \times 10^{-1}$	$5.88 \times 10^{-1}$
$\alpha_{42}$	$2.07 \times 10^{-4}$	$1.12 \times 10^{-3}$	$1.25 \times 10^{-3}$	$1.12 \times 10^{-3}$	$7.02 \times 10^{-3}$	444	113	29	14	8	6.4
$\alpha_7$	$1.95 \times 10^{-4}$	$3.02 \times 10^{-4}$	$2.72 \times 10^{-4}$	$3.02 \times 10^{-4}$	$4.84 \times 10^{-4}$	$7.57 \times 10^{-4}$	$3.02 \times 10^{-3}$	$1.2 \times 10^{-2}$	$2.72 \times 10^{-2}$	$4.83 \times 10^{-2}$	$6.27 \times 10^{-2}$
$\alpha_8$	$-2.06 \times 10^{-4}$	$-8.12 \times 10^{-4}$	$-2.43 \times 10^{-3}$	$-8.12 \times 10^{-4}$	$-3.25 \times 10^{-3}$	$-4.06 \times 10^{-3}$	$-8.12 \times 10^{-3}$	$-1.62 \times 10^{-2}$	$-2.43 \times 10^{-2}$	$-3.24 \times 10^{-2}$	$-3.7 \times 10^{-2}$
$\alpha_9$	$-2.83 \times 10^{-4}$	$-7.17 \times 10^{-4}$	$-2.37 \times 10^{-3}$	$-7.06 \times 10^{-4}$	$-1.67 \times 10^{-3}$	$-1.31 \times 10^{-3}$	$-5.91 \times 10^{-3}$	$-2.24 \times 10^{-2}$	$-9.3 \times 10^{-2}$	$-2.81 \times 10^{-1}$	$-1.5 \times 10^{-1}$
$\alpha_{51}$	$1.07 \times 10^{-4}$	$2.69 \times 10^{-4}$	$8.97 \times 10^{-4}$	$2.7 \times 10^{-4}$	$6.72 \times 10^{-4}$	$5.46 \times 10^{-4}$	$2.7 \times 10^{-3}$	$1.35 \times 10^{-2}$	$3.94 \times 10^{-2}$	$6.7 \times 10^{-2}$	$0.589 \times 10^{-1}$



TABLE II  
Data from Equations (73), (74), and (75)  
( $-1.91 < \rho < 0$ )

$\rho$	-1.90	-1.4	-1	-8	-4	-1	-0.1	-10 <sup>-3</sup>	-10 <sup>-4</sup>	-10 <sup>-5</sup>	-10 <sup>-6</sup>	-10 <sup>-7</sup>	-10 <sup>-8</sup>	-10 <sup>-9</sup>
$\alpha$	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>	-1.9bcd10 <sup>-12</sup>
$\epsilon$	6.9cd10 <sup>-10</sup>	2.07bcd10 <sup>-9</sup>	5.66bcd10 <sup>-9</sup>	9.28bcd10 <sup>-9</sup>	3.43bcd10 <sup>-8</sup>	4.2bcd10 <sup>-8</sup>	3.77cd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>	3.7bcd10 <sup>-8</sup>
$\alpha_{max}$	-6.8	-32.4	-53.4	-69	-1.3bcd10 <sup>-2</sup>	-4.7bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>	-4.4bcd10 <sup>-2</sup>
$\alpha_{min}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\alpha_1$	-6.35bcd10 <sup>-10</sup>	-3.45bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.12bcd10 <sup>-10</sup>	-2.82bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>	-1.7bcd10 <sup>-10</sup>
$\alpha_2$	-9.04bcd10 <sup>-10</sup>	-6.65bcd10 <sup>-10</sup>	-4.7bcd10 <sup>-10</sup>	-3.02bcd10 <sup>-10</sup>	-1.9bcd10 <sup>-10</sup>	-4.7bcd10 <sup>-10</sup>	-4.79bcd10 <sup>-10</sup>	-4.83bcd10 <sup>-10</sup>	-5.97bcd10 <sup>-10</sup>	-1.69bcd10 <sup>-10</sup>	-1.2bcd10 <sup>-10</sup>	-1.2bcd10 <sup>-10</sup>	-1.2bcd10 <sup>-10</sup>	-1.2bcd10 <sup>-10</sup>
$\epsilon_2$	1.18bcd10 <sup>-10</sup>	1.03bcd10 <sup>-10</sup>	1.00bcd10 <sup>-10</sup>	1.04bcd10 <sup>-10</sup>	1.52bcd10 <sup>-10</sup>	5.37bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>	5.33	5.33bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>	5.33bcd10 <sup>-10</sup>
$\alpha_{max}$	-1.42bcd10 <sup>-10</sup>	-1.92bcd10 <sup>-10</sup>	-2.7bcd10 <sup>-10</sup>	3.30bcd10 <sup>-10</sup>	-6.74bcd10 <sup>-10</sup>	-2.7bcd10 <sup>-10</sup>	-2.7bcd10 <sup>-10</sup>	-2.7bcd10 <sup>-10</sup>	-3.33bcd10 <sup>-10</sup>	-9.57bcd10 <sup>-10</sup>	-7.15bcd10 <sup>-10</sup>	-6.9bcd10 <sup>-10</sup>	-6.83bcd10 <sup>-10</sup>	-6.83bcd10 <sup>-10</sup>
$\alpha_3$	1.09bcd10 <sup>-10</sup>	5.93bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	9.68bcd10 <sup>-10</sup>	4.84bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>	3.02bcd10 <sup>-10</sup>
$\alpha_4$	1.54bcd10 <sup>-10</sup>	1.13bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	6.5bcd10 <sup>-10</sup>	3.55bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>	8.12bcd10 <sup>-10</sup>
$\epsilon_3$	2.5bcd10 <sup>-10</sup>	5.66bcd10 <sup>-10</sup>	1.10bcd10 <sup>-10</sup>	2.09bcd10 <sup>-10</sup>	2.93bcd10 <sup>-10</sup>	8.37bcd10 <sup>-10</sup>	7.3cd10 <sup>-10</sup>	7.19cd10 <sup>-10</sup>	7.3cd10 <sup>-10</sup>	7.19cd10 <sup>-10</sup>	7.18	7.18	7.18	7.18
$\alpha_{max}$	-1.43bcd10 <sup>-10</sup>	-1.92bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-6.74bcd10 <sup>-10</sup>	-6.74bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>	-2.63bcd10 <sup>-10</sup>
$\alpha_{min}$	-0.02	-5	-1.36	-3.2	-9.2	-1.03bcd10 <sup>-10</sup>	-8.93bcd10 <sup>-10</sup>	-8.93bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>	-9.02bcd10 <sup>-10</sup>

Imaginary



TABLE III  
Data from Equations (77), (78), and (79)  
( $\beta > 2.55 \times 10^5$ )

$\beta$	$10^{-5}$	$10^{-3}$	$10^{-2}$	$10^{-1}$	.68	1	2	10	$10^4$	$10^7$
$\alpha$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$	$2.49 \times 10^{10}$
$\zeta$	$1.08 \times 10^7$	$1.08 \times 10^7$	$1.08 \times 10^7$	$9.55 \times 10^6$	$5.60 \times 10^6$	$7.00 \times 10^6$	$8.00 \times 10^6$	$6.72 \times 10^6$	$5.74 \times 10^6$	$5.62 \times 10^6$
$\kappa_{\text{Re}}$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$
$\kappa_{\text{Im}}$	0	0	0	0	0	5.68	5.67	5.2	4.8	4.75
$\alpha$	$-2.22 \times 10^{10}$	$-2.22 \times 10^{10}$	$-2.22 \times 10^{10}$	$-2.22 \times 10^{10}$	$-1.02 \times 10^{10}$	$-2.22 \times 10^{10}$	$-8.86 \times 10^9$	$-2.22 \times 10^{10}$	$-2.22 \times 10^{10}$	$-2.22$
$\beta$	$-3.88 \times 10^{10}$	$-2.42 \times 10^{10}$	$-2.47 \times 10^{10}$	$-2.48 \times 10^{10}$	$-1.68 \times 10^{10}$	$-2.48 \times 10^{10}$	$-4.96 \times 10^9$	$-2.48 \times 10^{10}$	$-2.48 \times 10^{10}$	$-6.91$
$\zeta$	-4.13	$-4.13 \times 10^{-2}$	$-4.13 \times 10^{-2}$	$-4.13 \times 10^{-2}$	$-7.76 \times 10^{-2}$	$-6.61 \times 10^{-2}$	$-7.02 \times 10^{-2}$	$-2.52 \times 10^{-2}$	$-2.48 \times 10^{-2}$	-4.69
$\kappa_{\text{Re}}$	$1.08 \times 10^8$	$1.12 \times 10^7$	$1.12 \times 10^7$	$1.12 \times 10^7$	$1.65 \times 10^7$	$1.12 \times 10^7$	$5.58 \times 10^6$	$1.12 \times 10^7$	$1.12 \times 10^7$	2.11
$\kappa_{\text{Im}}$	$1.75 \times 10^6$	$1.67 \times 10^6$	$1.67 \times 10^6$	$1.67 \times 10^6$	4.64	2.68	1.41	1.01	1+	1+
$\alpha$	$7.32 \times 10^{10}$	$7.32 \times 10^{10}$	$7.32 \times 10^{10}$	$7.32 \times 10^{10}$	$3.38 \times 10^{10}$	$7.32 \times 10^{10}$	$2.93 \times 10^{10}$	$7.32 \times 10^{10}$	$7.32 \times 10^{10}$	7.32
$\beta$	$-8.22 \times 10^{10}$	$-8.22 \times 10^{10}$	$-8.22 \times 10^{10}$	$-8.22 \times 10^{10}$	$-5.6 \times 10^{10}$	$-8.22 \times 10^{10}$	$-1.64 \times 10^{10}$	$-8.22 \times 10^{10}$	$-8.22 \times 10^{10}$	$-2.28 \times 10^7$
$\zeta$	$-1.57 \times 10^{-1}$	$-1.57 \times 10^{-1}$	$-1.56 \times 10^{-1}$	$-1.39 \times 10^{-1}$	$-2.10 \times 10^{-1}$	$-1.07 \times 10^{-1}$	$-9.15 \times 10^{-2}$	$-9.83 \times 10^{-2}$	$-8.80 \times 10^{-2}$	$-1.55 \times 10^{-1}$
$\kappa_{\text{Re}}$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$	$\infty$
$\kappa_{\text{Im}}$	$2.13 \times 10^{10}$	$1.12 \times 10^8$	$1.12 \times 10^7$	$1.12 \times 10^6$	$1.65 \times 10^{10}$	$1.12 \times 10^{10}$	$5.58 \times 10^{10}$	$1.12 \times 10^{10}$	$1.12 \times 10^{10}$	3.02
$\kappa_{\text{Re}}$	$\infty$	$\infty$	$\infty$	$\infty$	$3.75 \times 10^4$	1.3	1.3	1.2	1.02	0.01
$\kappa_{\text{Im}}$	$\infty$	$\infty$	$\infty$	$\infty$	0	0	0	0	0	0





TABLE IV  
Data from Equations (77), (78), and (79)  
( $\beta < 0$ )

$\beta$	$-10^{-5}$	$-10^{-3}$	$-10^{-1}$	-2	-2.8	-10	$-10^2$	$-10^3$
$\alpha$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$	$2.49 \times 10^{-10}$
$\zeta$	$1.08 \times 10^{-1}$	$1.08 \times 10^{-1}$	$1.16 \times 10^{-1}$	$3.15 \times 10^{-1}$	$-8.67 \times 10^{-1}$	$-4.4 \times 10^{-1}$	$-5.50 \times 10^{-1}$	$-5.60 \times 10^{-1}$
$\alpha_{\text{max}}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\alpha_{\text{min}}$	0	0	0	3.36	-5.9	-4.21	-4.70	-4.75
$\alpha'$	$-2.2 \times 10^{-10}$	$-2.2 \times 10^{-10}$	$-2.2 \times 10^{-10}$	$-8.86 \times 10^{-10}$	$-1.74 \times 10^{-9}$	$-2.2 \times 10^{-8}$	$-2.2 \times 10^{-6}$	$-2.2 \times 10^{-5}$
$\alpha'_1$	$-8.84 \times 10^{-10}$	$-2.54 \times 10^{-9}$	$-2.48 \times 10^{-8}$	$-4.96 \times 10^{-7}$	$-6.94 \times 10^{-7}$	$-2.48 \times 10^{-5}$	$-2.48 \times 10^{-3}$	$-2.44 \times 10^{-2}$
$\zeta_1$	4.13	$4.13 \times 10^2$	$4.16 \times 10^4$	$7.02 \times 10^5$	$8.42 \times 10^5$	$2.44 \times 10^7$	$2.48 \times 10^9$	$2.46 \times 10^{10}$
$\alpha_{\text{max}}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\alpha_{\text{min}}$	$-4.38 \times 10^{-10}$	$-1.15 \times 10^{-9}$	$-1.12 \times 10^{-6}$	$-5.60 \times 10^{-5}$	$-4.00 \times 10^{-4}$	$-1.12 \times 10^{-2}$	$-1.12 \times 10^3$	$-1.12 \times 10^2$
$\alpha'_2$	$7.32 \times 10^{-10}$	$7.32 \times 10^{-6}$	$7.32 \times 10^{-2}$	$2.93 \times 10^{-1}$	$5.73 \times 10^{-1}$	$7.32 \times 10^1$	$7.32 \times 10^5$	$7.32 \times 10^9$
$\alpha'_3$	$8.22 \times 10^{-10}$	$8.22 \times 10^{-8}$	$8.22 \times 10^{-6}$	$1.64 \times 10^{-5}$	$2.30 \times 10^{-5}$	$8.22 \times 10^{-4}$	$8.21 \times 10^{-2}$	$8.08 \times 10^{-2}$
$\zeta_2$	$1.57 \times 10^{-1}$	$1.57 \times 10^{-1}$	$1.75 \times 10^3$	$9.15 \times 10^5$	$3.50 \times 10^6$	$-6.29 \times 10^7$	$-8.04 \times 10^9$	$-8.13 \times 10^{12}$
$\alpha_{\text{max}}$	$-1.10 \times 10^{-1}$	$-1.12 \times 10^{-1}$	$-5.60 \times 10^{-1}$	$-5.60 \times 10^{-1}$	$-4.00 \times 10^{-1}$	$-1.12 \times 10^{-1}$	$-1.12 \times 10^3$	$-1.12 \times 10^2$
$\alpha_{\text{min}}$	$-1.98 \times 10^{-6}$	$-2.12 \times 10^{-6}$	$-2.12 \times 10^{-4}$	5.57	$-1.52 \times 10^{-4}$	0	0	0



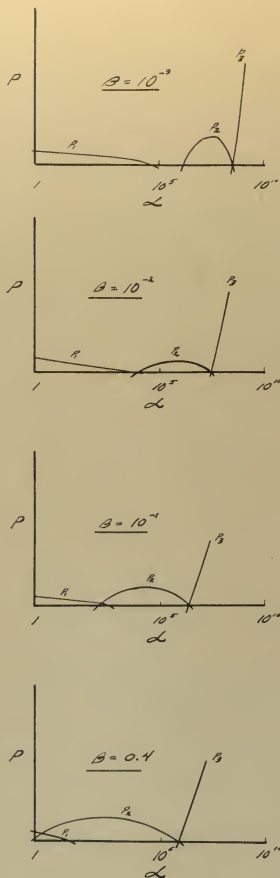


Figure 6. Variation of  $p_1$ ,  $p_2$ , and  $p_3$  with  $\alpha$  for several values of positive  $\beta$ . ( $\frac{m_1}{m_2} = 7.75$ )



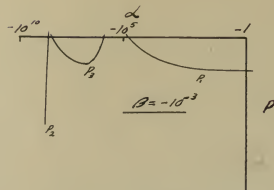
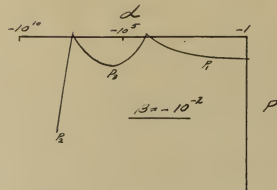
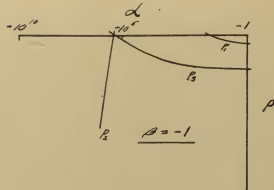
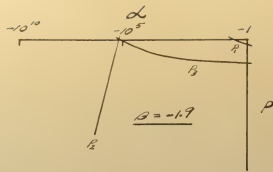


Figure 7. Variation of  $p_1$ ,  $p_2$ , and  $p_3$  with  $\alpha$  for several values of negative  $B$ . ( $\frac{M_1}{M_2} = 7.75$ )



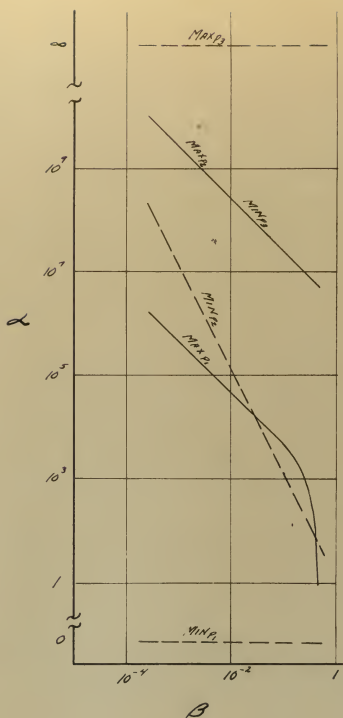


Figure 8. Minimum and maximum limits of  $\alpha$  which yield positive  $p_i$  for positive  $\beta$ . ( $\frac{M_1}{M_2} = 7.75$ )





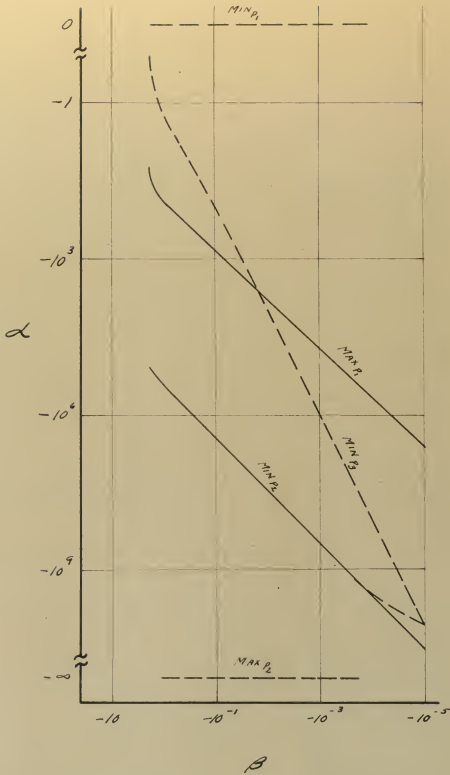


Figure 9. Minimum and maximum limits of  $\alpha$  which yield negative  $p_i$  for negative  $\beta$ . ( $\frac{M_1}{M_2} = 7.75$ )



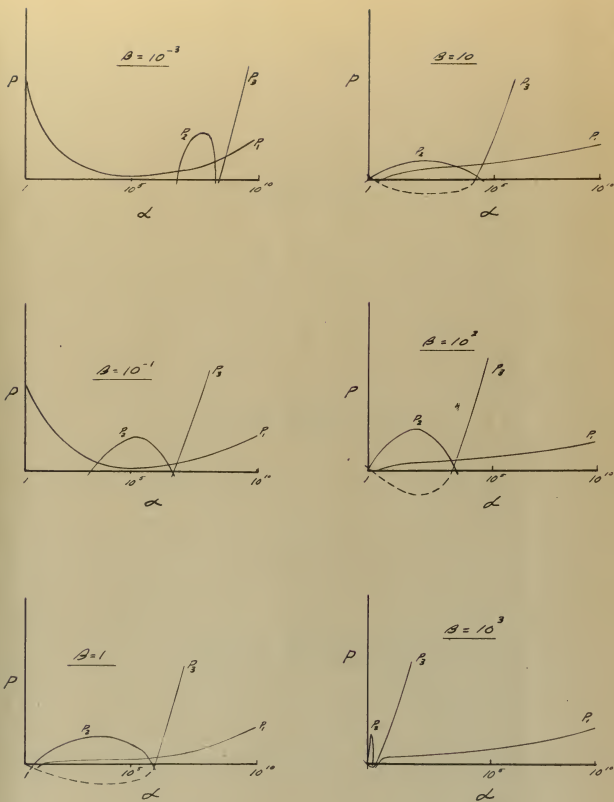


Figure 10. Variation of  $p_1$ ,  $p_2$ , and  $p_3$  with  $\alpha$  for several values of positive  $\beta$ . ( $\frac{m_1}{m_2} \approx 1.5$ )



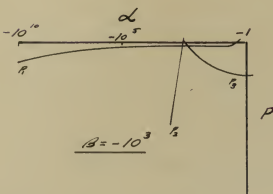
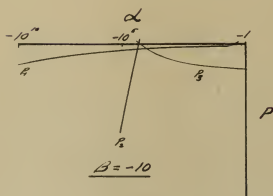
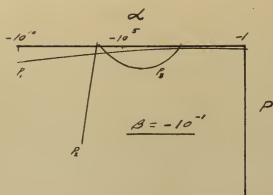
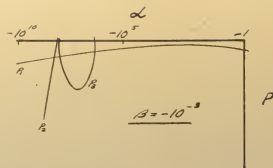


Figure 11. Variation of  $p_i$ ,  $p_k$ , and  $p_s$  with  $\alpha$  for several values of negative  $\beta$ . ( $\frac{M_1}{M_2} = 1.5$ )



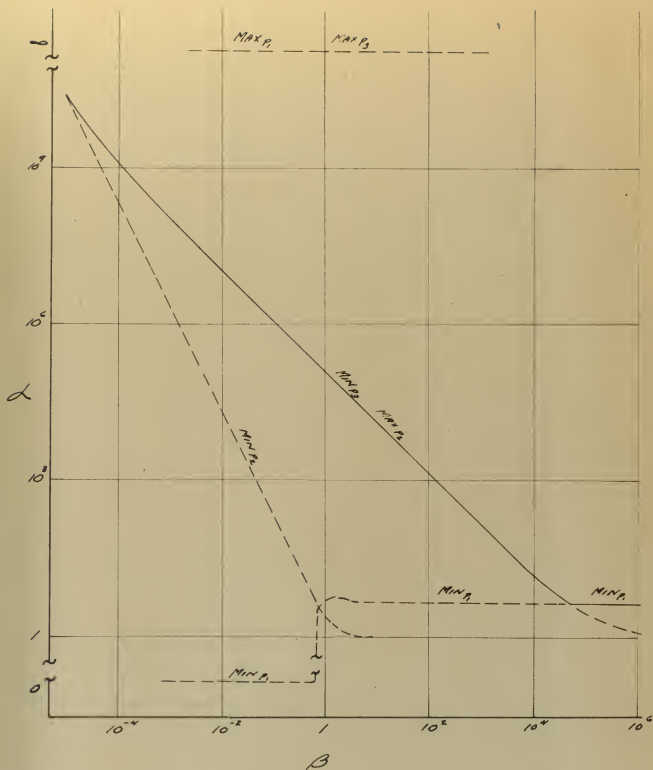


Figure 12. Minimum and maximum limits of  $\alpha$  which yield positive  $p_i$  for positive  $\beta$ . ( $\frac{M_1}{M_2} = 1.5$ )





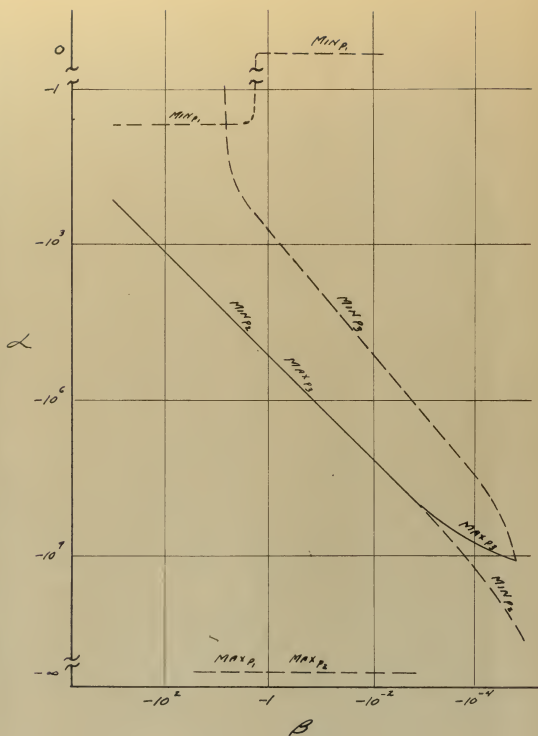


Figure 13. Minimum and maximum limits of  $\alpha$  which yield negative  $p_\alpha$  for negative  $\beta$ . ( $\frac{M_1}{M_2} = 1.5$ )



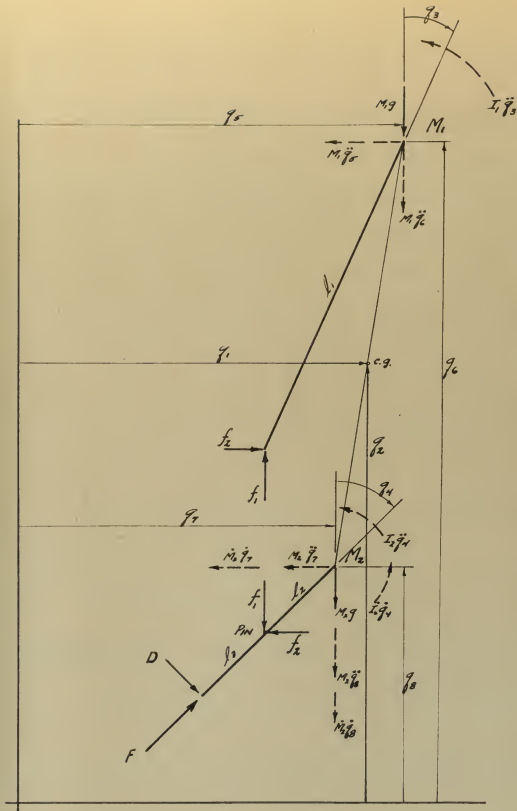


Figure 14. Schematic diagram of system showing inertia forces for use in Newtonian derivation of equations of motion.



# APPENDIX

## NEWTONIAN DERIVATION OF EQUATIONS OF MOTION

The equations of motion for the two dimensional system may be derived by applying Newton's Second Law of Motion provided that proper consideration is given to the inertia forces of the system. In Figure 14 these forces are indicated by broken lines. In this derivation the main rocket,  $M_1$ , and the booster,  $M_2$ , will be treated separately as free bodies and the reaction of the pin connection will be replaced in Figure 14 by its two components,  $f_1$  in the vertical direction, and  $f_2$  in the horizontal direction. The pin is assumed to be frictionless and therefore no moment is transmitted.

Treating the main rocket,  $M_1$ , as a free body first, the sum of the vertical forces acting on it are

$$\Sigma V = f_1 - M_1 g - M_1 \ddot{y}_1$$

and therefore

$$f_1 = M_1 g + M_1 \ddot{y}_1 \quad (1)$$

Similarly, the sum of the horizontal forces acting on  $M_1$  are

$$\Sigma H = f_2 - M_1 \ddot{x}_1$$

and therefore

$$f_2 = M_1 \ddot{x}_1 \quad (2)$$



Finally, the sum of the moments acting on  $M_1$  are

$$\Sigma M = I_1 \ddot{\theta}_3 + f_2 l_1 \cos \theta_3 - f_1 l_1 \sin \theta_3$$

or

$$I_1 \ddot{\theta}_3 = f_1 l_1 \sin \theta_3 - f_2 l_1 \cos \theta_3 \quad (3)$$

Treating the booster,  $M_2$ , as a free body, the sum of the vertical forces acting on it are

$$\Sigma V = F \cos \theta_4 - D \sin \theta_4 - f_1 - M_2 g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8$$

and therefore

$$f_1 = F \cos \theta_4 - D \sin \theta_4 - M_2 g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8 \quad (4)$$

Furthermore, the sum of the horizontal forces acting on  $M_2$  are

$$\Sigma H = F \sin \theta_4 + D \cos \theta_4 - f_2 - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7$$

and therefore

$$f_2 = F \sin \theta_4 + D \cos \theta_4 - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7 \quad (5)$$

Finally, the sum of the moments acting on  $M_2$  are

$$\Sigma M = I_2 \ddot{\theta}_4 + I_2 \ddot{\theta}_4 + f_1 l_2 \sin \theta_4 - f_2 l_2 \cos \theta_4 + D(l_2 + l_3)$$

or

$$I_2 \ddot{\theta}_4 + I_2 \ddot{\theta}_4 = f_2 l_2 \cos \theta_4 - f_1 l_2 \sin \theta_4 - D(l_2 + l_3) \quad (6)$$

Substituting the results of Equation (1) into Equation (4) to eliminate  $f_1$ ,

$$M_1 g + M_1 \ddot{\theta}_6 = F \cos \theta_4 - D \sin \theta_4 - M_2 g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8 \quad (7)$$

Similarly, substituting the results of Equation (2) into Equation (5) to eliminate  $f_2$

$$M_1 \ddot{\theta}_5 = F \sin \theta_4 + D \cos \theta_4 - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7 \quad (8)$$





Substituting the results of Equations (4) and (5) into Equation (3) to eliminate  $f_1$  and  $f_2$

$$\begin{aligned} I_1 \ddot{\theta}_2 &= (F \cos \theta_4 - D \sin \theta_4 - M_2 g - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_8) l_1 \sin \theta_3 \\ &\quad - (F \sin \theta_4 + D \cos \theta_4 - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7) l_1 \cos \theta_3 \end{aligned} \quad (9)$$

Similarly, substituting the results of Equations (4) and (5) into Equation (6) to eliminate  $f_1$  and  $f_2$

$$\begin{aligned} I_2 \ddot{\theta}_4 + \dot{I}_2 \dot{\theta}_4 &= (F \sin \theta_4 + D \cos \theta_4 - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7) l_2 \cos \theta_4 \\ &\quad - (F \cos \theta_4 - D \sin \theta_4 - M_2 g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8) l_2 \sin \theta_4 \\ &\quad - D(l_2 + l_3) \end{aligned} \quad (10)$$

Assuming the angles  $q_3$  and  $q_4$  sufficiently small such that terms of order  $q^4$  may be neglected, the following approximations can be made

$$\sin \theta_i \approx \theta_i \quad \text{and} \quad \cos \theta_i \approx 1$$

Therefore, Equations (7), (8), (9), and (10) may be rewritten in simpler form. Thus

From Equation (7)

$$M_1 \ddot{\theta}_2 = F - D g_4 - (M_1 + M_2) g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8 \quad (11)$$

From Equation (8)

$$M_1 \ddot{\theta}_5 = F g_4 + D - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7 \quad (12)$$

From Equation (9)

$$\begin{aligned} I_1 \ddot{\theta}_2 &= (F - D g_4 - M_2 g - M_2 \ddot{\theta}_8 - \dot{M}_2 \dot{\theta}_8) l_1 \theta_3 \\ &\quad - (F g_4 + D - M_2 \ddot{\theta}_7 - \dot{M}_2 \dot{\theta}_7) l_1 \end{aligned} \quad (13)$$



From Equation (10)

$$L_2 \ddot{q}_4 + L_2 \dot{q}_4 = (F q_4 + D - M_2 \ddot{q}_1 - M_2 \dot{q}_1) L_2 - (F - D q_4 - M_2 g - M_2 \ddot{q}_3 - M_2 \dot{q}_3) L_2 q_4 - D(L_1 + L_2) \quad (14)$$

The following approximate relations between the principal coordinates  $q_1, q_2, q_3$ , and  $q_4$ , and the extraneous coordinates  $q_5, q_6, q_7, q_8$  were proved earlier in Part I of this paper

$$q_5 \approx q_1 + \frac{M_2}{M_1 + M_2} (L_1 q_3 - L_2 q_4) \quad (15)$$

$$q_6 \approx q_2 + \frac{M_2}{M_1 + M_2} (L_1 - L_2) \quad (16)$$

$$q_7 \approx q_1 - \frac{M_1}{M_1 + M_2} (L_1 q_3 - L_2 q_4) \quad (17)$$

$$q_8 \approx q_2 - \frac{M_1}{M_1 + M_2} (L_1 - L_2) \quad (18)$$

Taking the first and second time derivatives of Equation (15)

$$\dot{q}_5 = \dot{q}_1 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (L_1 q_3 - L_2 q_4) + \frac{M_2}{M_1 + M_2} (L_1 \dot{q}_3 - L_2 \dot{q}_4) \quad (19)$$

and

$$\begin{aligned} \ddot{q}_5 = \ddot{q}_1 - L \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^3} (L_1 q_3 - L_2 q_4) + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (L_1 \dot{q}_3 - L_2 \dot{q}_4) \\ + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (L_1 \ddot{q}_3 - L_2 \ddot{q}_4) + \frac{M_2}{M_1 + M_2} (L_1 \ddot{q}_3 - L_2 \ddot{q}_4) \end{aligned} \quad (20)$$

Taking the first and second time derivatives of Equation (16)

$$\dot{q}_6 = \dot{q}_2 + \frac{M_1 \dot{M}_2}{(M_1 + M_2)^2} (L_1 - L_2) \quad (21)$$

and

$$\ddot{q}_6 = \ddot{q}_2 - L \frac{M_1 \dot{M}_2^2}{(M_1 + M_2)^3} (L_1 - L_2) \quad (22)$$

Taking the first and second time derivatives of Equation (17)



$$\dot{g}_1 = \dot{g}_1 + \frac{M_1 \dot{r}_2}{(M_1 + M_2)^2} (l_1 \dot{g}_2 - l_2 \dot{g}_4) - \frac{M_1}{M_1 + M_2} (l_1 \dot{g}_2 - l_2 \dot{g}_4) \quad (23)$$

and

$$\begin{aligned} \ddot{g}_1 = \ddot{g}_1 - 2 \frac{M_1 \dot{r}_2^2}{(M_1 + M_2)^3} (l_1 \dot{g}_2 - l_2 \dot{g}_4) + \frac{M_1 \ddot{r}_2}{(M_1 + M_2)^3} (l_1 \dot{g}_2 - l_2 \dot{g}_4) \\ + \frac{M_1 \dot{r}_2}{(M_1 + M_2)^2} (l_1 \ddot{g}_2 - l_2 \ddot{g}_4) - \frac{M_1}{M_1 + M_2} (l_1 \ddot{g}_2 - l_2 \ddot{g}_4) \end{aligned} \quad (24)$$

Taking the first and second time derivatives of Equation (18)

$$\dot{g}_2 = \dot{g}_2 + \frac{M_1 \dot{r}_2}{(M_1 + M_2)^2} (l_1 - l_2) \quad (25)$$

and

$$\ddot{g}_2 = \ddot{g}_2 - 2 \frac{M_1 \dot{r}_2^2}{(M_1 + M_2)^3} (l_1 - l_2) \quad (26)$$

Equation (11) can now be rewritten in terms of the principal coordinates only by substituting the results of Equations (22), (25) and (26) for the extraneous coordinates

$$(M_1 + M_2) \ddot{g}_2 - \frac{M_1 \dot{r}_2^2}{(M_1 + M_2)^2} (l_1 - l_2) + \dot{r}_2 \dot{g}_2 = F - g(M_1 + M_2) - D \dot{g}_4 \quad (27)$$

Note that Equation (27) is identical with Equation (27) of Part II of this paper.

In a similar manner, Equation (12) can be rewritten in terms of the principal coordinates by substituting the results of Equations (20), (23), and (24) for the extraneous coordinates

$$(M_1 + M_2) \ddot{g}_1 + \frac{M_1 \dot{r}_2}{M_1 + M_2} (l_1 \dot{g}_2 - l_2 \dot{g}_4) - \frac{M_1 \dot{r}_2^2}{(M_1 + M_2)^2} (l_1 \dot{g}_2 - l_2 \dot{g}_4) + \dot{r}_2 \dot{g}_1 = F \dot{g}_4 + D \quad (28)$$

which is identical with Equation (23) of Part II of this paper.

Substitution of Equations (23), (24), (25), and (26) in Equation (13) yields



$$\begin{aligned}
 & I_1 \ddot{\theta}_3 + \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_1 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) + \frac{M_1 M_2}{M_1 + M_2} l_1 (l_1 \ddot{\theta}_3 - l_2 \ddot{\theta}_4) - \frac{M_1 \dot{\theta}_3^2}{M_1 + M_2} l_1 \dot{\theta}_1 \\
 & - \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_1 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) + \frac{M_1 \dot{\theta}_3^2}{M_1 + M_2} l_1 \dot{\theta}_3 \dot{\theta}_2 + \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_1 \dot{\theta}_3 (l_1 - l_2) \\
 & = - \frac{M_1 l_1}{M_1 + M_2} [F(\theta_4 - \theta_3) + D]
 \end{aligned} \tag{29}$$

which is identical with Equation (44) of Part II of this paper.

Finally, substitution of Equations (23), (24), (25), and (26)

in Equation (14) yields

$$\begin{aligned}
 & I_2 \ddot{\theta}_4 + l_2 \dot{\theta}_4^2 - \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_2 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) - \frac{M_1 M_2}{M_1 + M_2} l_2 (l_1 \ddot{\theta}_3 - l_2 \ddot{\theta}_4) \\
 & + \frac{M_1 \dot{\theta}_3^2}{M_1 + M_2} l_2 \dot{\theta}_1 + \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_2 (l_1 \dot{\theta}_3 - l_2 \dot{\theta}_4) - \frac{M_1 \dot{\theta}_3^2}{M_1 + M_2} l_2 \dot{\theta}_4 \dot{\theta}_2 \\
 & - \frac{M_1^2 \dot{\theta}_3^2}{(M_1 + M_2)^2} l_2 \dot{\theta}_4 (l_1 - l_2) = -D \left[ \frac{M_2}{M_1 + M_2} l_2 + l_3 \right]
 \end{aligned} \tag{30}$$

which is identical with Equation (59) of Part II of this paper.





TABLE OF SYMBOLS

$\alpha$	- $\frac{h_1}{h_2}$
$\beta$	- $\frac{h_2}{k_2}$
D	- Jet damping force
F	- Thrust force
g	- Acceleration of gravity
I	- Moment of inertia
$I_{sp}$	- Specific Impulse
k	- Radius of gyration
M	- Mass
$\dot{M}$	- Mass rate of flow
q	- Lagrangian generalized space coordinate
Q	- Dimensionless Lagrangian generalized space coordinate
$\mathcal{Q}$	- Lagrangian generalized force
T	- Kinetic energy



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